

OPTIMAL QUADRATURE FORMULA OF THE HERMITE TYPE

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Abstract: In this work, we discuss the construction of derivative optimal quadrature formulas in the Sobolev space. We derive an analytical expression for an error functional norm and obtain a system of linear equations Winner-Hopf type by the coefficients using the method of Lagrange multipliers. Further, we apply the Sobolev method to get the analytical representation of the optimal coefficients. Using these coefficients, we calculate the norm of the error functional of the optimal quadrature formula in cases $m=3$ and $m=4$ in Sobolev $L_2^{(m)}(0,1)$ space. We verify our theoretical conclusions with the help of numerical experiments.

Keywords: Error functional, extremal function, Sobolev space, optimal coefficients, optimal quadrature formula.

INTRODUCTION

Currently, there are various approaches to constructing quadrature formulas. One of them is the classical approach to constructing quadrature formulas, which requires the accuracy of the polynomial formula for the highest possible order. Another of them is the functional approach to constructing quadrature formulas. In this case, the quadrature formula is designed to minimize the error functional norm in a given Banach space [1,2,3,4]. Article [5] presents new and effective quadrature formulas, which merge function and first derivative estimation at equally-spaced data points, with a particular emphasis on improving computational efficiency in terms of both cost and time. The objective of the research presented in work [6] is to simplify the computation of the components involved in the integral transformation, denoted as F^m and $m=0$. The analytical expressions for these components encompass definite integrals. Instead of the Newton-Cotes formulas, it is proposed to use non-trivial quadrature formulas with unevenly distributed integration points. The quadrature method is essential in the approximate solution of integral equations. In [7], the trapezoidal numerical integration formula is used to solve the Fredholm-Hammerstein integral equations. In [8], the perturbed Milne quadrature rule was derived for n -fold differentiable functions. The following articles [9-16] create optimal quadrature formulas for different Hilbert spaces. Additionally, precise estimates of the designed recipes are provided.

The work consists of the following sections: In the first Section, we introduce the introduction; In the second Section, the statement of the problem of constructing a quadrature formula is considered; In the third Section, the general form of the error functional norm is derived using the extremal function. In Section 4, we derived a system of linear equations of the Wiener-Hopf type for the coefficients of the quadrature formula being considered in the space $L_2^{(m)}(0,1)$. Additionally, we found the analytical form of the optimal coefficients of the quadrature formulas (1) for $m=3$ and $m=4$ in the space $L_2^{(m)}(0,1)$.

STATEMENT OF THE PROBLEM

We consider the following quadrature formula

$$\int_0^1 f(x) dx \cong \sum_{\beta=0}^N K_0[\beta] f(h\beta) + \frac{h^2}{12} (f'(0) - f'(1)) + \sum_{\beta=0}^N K_1[\beta] f''(h\beta) \quad (1)$$

where $K_0[\beta] = \frac{h}{2}$, $\beta = 0, N$, $\frac{h}{2}$, $\beta = \overline{1, N-1}$, and $K_1[\beta]$ are the unknown coefficients of the quadrature formula

(1), $f \in L_2^{(m)}(0,1)$, $L_2^{(m)}(0,1)$ is the space of functions which are square integrable with m -th generalized derivative.

The difference between the exact value and the approximated value obtained from the quadrature formula is referred to as the error of the formula.

$$\begin{aligned} & \int_0^1 f(x) dx - \sum_{\beta=0}^N K_0[\beta] f(h\beta) - \frac{h^2}{12} (f'(0) - f'(1)) - \sum_{\beta=0}^N K_1[\beta] f''(h\beta) \\ & = \ell_N(x) f(x) dx = (\ell_N, f). \end{aligned} \quad (2)$$

From difference (2) it follows that quadrature formula (1) corresponds to the following error functional

$$\ell_N(x) = \chi_{[0,1]}(x) - \sum_{\beta=0}^N K_0[\beta] \delta(x - h\beta) + \frac{h^2}{12} (\delta'(x) - \delta'(x-1)) - \sum_{\beta=0}^N K_1[\beta] \delta''(x - h\beta) \quad (3)$$

where, $\delta(x)$ is the Dirac's delta-function, and $\chi_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$.

The norm in space $L_2^{(m)}(0,1)$ is defined by the following form:

$$\|f\|_{L_2^{(m)}(0,1)}^2 = \int_0^1 (f^{(m)}(x))^2 dx.$$

In addition, the error functional (3) is required to satisfy the following conditions

$$(\ell_N(x), x^\alpha) = 0, \quad \alpha = 0, 1, 2, \dots, m-1. \quad (4)$$

Error (2) based on the Cauchy-Schwartz inequality is estimated as follows

$$|(\ell_N, f)| \leq \| \ell_N \|_{L_2^{(m)*}(0,1)} \| f \|_{L_2^{(m)}(0,1)}. \quad (5)$$

Here, we have been receiving the following tasks.

1. Find the norm of the error functional (3) of the quadrature formula (1).
2. Finding the coefficients $K_1[\beta]$, $\beta = \overline{0, N}$, which give the smallest value to the norm $\| \ell_N \|_{L_2^{(m)*}}$ of the error functional (3), that is, calculating the value

$$\| \ell_N \|_{L_2^{(m)*}} = \inf_{K_1[\beta]} \| \ell_N \|_{L_2^{(m)*}}. \quad (6)$$

It is important to mention that, the coefficients $K_1[\beta]$, $\beta = \overline{0, N}$ satisfying the quantity (6) of the error functional (3), i.e., specifying the minimum norm $\| \ell_N \|_{L_2^{(m)*}}$. If such coefficients exist, these

are called optimal coefficients, denoted as $\hat{K}_1[\beta]$. To solve these problems, we must sequentially do the following:

a) find the general form of ℓ_N ;

b) seek the optimal coefficient $\hat{K}_1[\beta]$ that minimizes the norm of the error functional ℓ_N .

In the following section, our main objective is to determine the norm of the error functional.

FINDING AND MINIMIZATION OF THE NORM OF THE ERROR FUNCTIONAL

We will use Riesz's theorem on the general form of a linear continuous functional in a Hilbert space [17,1

$$(\ell_N, f) = \langle U_{\ell_N}, f \rangle$$

and

$$\|\ell_N\|_{L_2^{(m)*}} = \|U_{\ell_N}\|_{L_2^{(m)}},$$

where

$$\langle U_{\ell_N}, f \rangle = \int_0^1 U^{(m)}(x) f^{(m)}(x) dx.$$

This means that, to find the general form of the norm of the error functional (3), it is necessary to determine the function U_{ℓ_N} . To do this, we will use the concept of an extremal function introduced by S.L. Sobolev [1]. The extremal function U_{ℓ_N} corresponding to the error functional

ℓ_N defined by S.L. Sobolev on $L_2^{(m)}(0,1)$ space

$$U_{\ell_N}(x) = (-1)^m \ell_N(x) * \mu_m(x) + P_{m-1}(x). \quad (7)$$

where

$$\mu_m(x) = \frac{|x|^{2m-1}}{2(2m-1)!}$$

and this is the solution to equation $\frac{d^{2m}}{dx^{2m}} \mu_m(x) = \delta(x)$ [1], $P_{m-1}(x)$ degree polynomial $m-1$ and $*$ is the action of convolution.

Recall that the convolution of two functions φ and g is defined as follows [1]

$$\varphi(x) * g(x) = \int_{-\infty}^{+\infty} \varphi(x-y)g(y)dy = \int_{-\infty}^{+\infty} \varphi(y)g(x-y)dy.$$

The following equality is appropriate for the error functional ℓ_N norm in space $L_2^{(m)*}(0,1)$ [1].

$$\|\ell_N\|_{L_2^{(m)*}}^2 = (\ell_N, U_{\ell_N}) = (\ell_N(x), (-1)^m \ell_N(x) * \mu_m(x))$$

Having performed some calculations on the right side of the last equality, we take the general form of the squared norm of ℓ_N corresponding to the quadrature formula (1)

$$\|\ell_N\|_{L_2^{(m)*}(0,1)}^2 = (\ell_N, U_{\ell_N}) = \int_{-\infty}^{+\infty} \ell_N(x) U_{\ell_N}(x) dx$$

$$\begin{aligned}
 &= (-1)^m \sum_{\beta=0}^N K_1[\beta] \sum_{\gamma=0}^N K_1[\gamma] \frac{|h\beta - h\gamma|^{2m-5}}{2(2m-5)!} - 2 \sum_{\beta=0}^N K_1[\beta] \int_0^1 \frac{|x - h\beta|^{2m-3}}{2(2m-3)!} dx \\
 &- \frac{h^2}{6} \sum_{\beta=0}^N K_1[\beta] \frac{(h\beta)^{2m-4} + (h\beta-1)^{2m-4}}{2(2m-4)!} \\
 &+ 2 \sum_{\beta=0}^N K_0[\beta] \sum_{\gamma=0}^N K_1[\gamma] \frac{|h\beta - h\gamma|^{2m-3}}{2(2m-3)!} - \frac{h^2}{6} \sum_{\beta=0}^N K_0[\beta] \frac{(h\beta)^{2m-2} + (h\beta-1)^{2m-2}}{2(2m-2)!} \\
 &- 2 \sum_{\beta=0}^N K_0[\beta] \int_0^1 \frac{|x - h\beta|^{2m-1}}{2(2m-1)!} dx + \sum_{\beta=0}^N K_0[\beta] \sum_{\gamma=0}^N K_0[\gamma] \frac{|h\beta - h\gamma|^{2m-1}}{2(2m-1)!} \\
 &+ \frac{h^2}{6(2m-1)!} + \frac{1}{(2m+1)!} + \frac{h^4}{144(2m-3)!}. \quad (8)
 \end{aligned}$$

So, problem 1 is completely solved.

TO FIND THE COEFFICIENTS OF OPTIMAL QUADRATURE FORMULA

Let us determine the coefficients $\dot{K}_1[\beta]$. To do this, we construct the following Lagrange function

$$\Psi(K_1[0], K_1[1], \dots, K_1[N], \lambda_0, \lambda_1, \dots, \lambda_{m-1})$$

$$= \|\ell_N\|_{L_2^{(m)*}}^2 - 2 \sum_{\alpha=0}^{m-1} (-1)^\alpha \lambda_\alpha (\ell_N, x^\alpha). \quad (9)$$

From the constructed Lagrange function (9), we get the partial derivatives by the coefficients $K_1[\beta]$, $(\beta = 0, \dots, N)$ and, λ_α , $(\alpha = 0, 1, \dots, m-1)$. As result of this we have the following system of linear equations respect to $K_1[\beta]$ and λ_α

$$\sum_{\gamma=0}^N K_1[\gamma] \mu_m^{(IV)}(h\beta - h\gamma) + P_{m-3}(h\beta) = F_m(h\beta), \quad \beta = \overline{0, N}, \quad (10)$$

$$\sum_{\gamma=0}^N K_1[\gamma] (h\gamma)^\alpha = - \sum_{j=1}^{\alpha} \frac{\alpha! B_{\alpha+3-j}}{j!(\alpha+3-j)!} h^{\alpha+3-j}, \quad \alpha = \overline{0, m-3}, \quad (11)$$

where $\mu_m^{(IV)}(x) = \mu_{m-2}(x) = \frac{|x|^{2m-5}}{2(2m-5)!}$, $P_{m-3}(h\beta)$ is an unknown polynomial of degree $m-3$ whose coefficients are expressed through λ_α , B_{i+3} are Bernoulli numbers, and

$$F_m(h\beta) = \sum_{i=0}^{2m-5} \frac{(h\beta)^{2m-5-i}}{(2m-5-i)!} - \frac{B_{i+3} h^{i+3}}{(i+3)!} + \sum_{j=1}^i \frac{(-1)^j B_{i+3-j}}{2j!(i+3-j)!} h^{i+3-j} + \frac{B_{2m-2} h^{2m-2}}{(2m-2)!}. \quad (12)$$

This system has a unique solution at every fixed N point and gives a minimum value to norm, $\|\ell_N\|_{L_2^{(m)*}}^2$ is a quadratic function of $K_1[\beta]$, $(\beta = 0, \dots, N)$ the coefficients of many variables. As a result, it reaches a single minimum at a specific value of $K_1[\beta] = \dot{K}_1[\beta]$. The ones found $\dot{K}_1[\beta]$ are called optimal coefficients and the corresponding quadrature formula is called optimal quadrature formula.

Theorem 1. For the coefficients of the optimal quadrature formula (1) in $L_2^{(3)}(0,1)$ Sobolev space, the following equalities hold:

$$K_1[\beta] = 0, \quad \beta = \overline{0, N}. \quad (13)$$

Theorem 2. For the coefficients of the optimal quadrature formula (1) in $L_2^{(4)}(0,1)$ Sobolev space the following equalities hold:

$$\begin{aligned} K_1[0] &= ah \frac{q - q^N}{q - 1}, \\ K_1[\beta] &= ah(q^\beta + q^{N-\beta}), \quad \beta = \overline{1, N-1}, \\ K_1[N] &= ah \frac{q - q^N}{q - 1}, \end{aligned} \quad (14)$$

where $a = \frac{h^2}{120(1+q^N)}$, $q = \sqrt{3} - 2$.

Here we are engaged in the proof of Theorem 2, and Theorem 1 is proved analogously.

Proof of Theorem 2. We can obtain the following system by using equations (10) through (12) from $m = 4$

$$K_1[\gamma] \mu_4^{(IV)}(h\beta - h\gamma) + P_1(h\beta) = F_4(h\beta), \quad \beta = \overline{0, N}, \quad (15)$$

$$K_1[\gamma] = 0, \quad (16)$$

$$K_1[\gamma](h\beta) = 0, \quad (17)$$

where $\mu_4^{(IV)}(h\beta) = \frac{|h\beta|^3}{12}$,

$$F_4(h\beta) = \frac{h^4}{1440} (h\beta)^2 - (h\beta) + \frac{1}{2} + \frac{h^6}{30240}. \quad (18)$$

Now we should solve the system of linear equations (10)-(11). To get an analytical solution for these equations, we need a discrete analogue of the differential operator $\frac{d^4}{dx^4}$ (see, [19]), i.e.,

$$D_2(h\beta) = \begin{cases} 6\sqrt{3}q^{|\beta|}, & |\beta| = 2, \\ \frac{6}{h^4} (19 - 12\sqrt{3}), & |\beta| = 1, \\ 6\sqrt{3} - 8, & \beta = 0. \end{cases} \quad (19)$$

here $q = \sqrt{3} - 2$, $\delta(h\beta) = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \neq 0 \end{cases}$

and the operator $D_2(h\beta)$ has the following properties

$$D_2(h\beta) * G_2(h\beta) = \delta(h\beta),$$

$$D_2(h\beta)(h\beta)^k = \begin{cases} 0, & 0 \leq k \leq 3, \\ (2m-4)!, & k = 4. \end{cases}$$

When $\beta < 0$ and $\beta > N$ let $K_1[\beta] = 0$. Then we write equation (15) in the form of convolution

$$K_1[\gamma] * \mu_4^{(IV)}(h\beta) + P_1(h\beta) = F_4(h\beta), \quad \beta = \overline{0, N}. \quad (20)$$

We will use the notation presented on the left-hand side of equation (20).

$$u(h\beta) = K_1[\beta] * \mu_4^{(IV)}(h\beta) + P_1(h\beta). \quad (21)$$

According to properties $D_2(h\beta)$ we have

$$\begin{aligned} K_1[\beta] &= hD_2[\beta] * u(h\beta) = hD_2(h\beta) * (\mu_4^{(IV)}(h\beta) * K_1[\beta] + P_1(h\beta)) \\ &= hK_1[\beta] * (D_2(h\beta) * \mu_4^{(IV)}(h\beta)) = hK_1[\beta] * \delta(h\beta) = K_1[\beta]. \end{aligned} \quad (22)$$

This means that to find the optimal coefficients $K_1[\beta]$ it is necessary to determine the values of the function $u(h\beta)$ for all integer values β .

for $\beta = 0, 1, \dots, N$ $u(h\beta) = F_4(h\beta)$.

for $\beta < 0$

$$\begin{aligned} u(h\beta) &= -\frac{(h\beta)^3}{12} \sum_{\gamma=0}^N K_1[\gamma] + \frac{(h\beta)^2}{4} \sum_{\gamma=0}^N K_1[\gamma](h\gamma) \\ &\quad - \frac{(h\beta)}{4} \sum_{\gamma=0}^N K_1[\gamma](h\gamma)^2 + \frac{1}{12} \sum_{\gamma=0}^N K_1[\gamma](h\gamma)^3 + P_1(h\beta) = Q_1^-(h\beta). \end{aligned} \quad (23)$$

According to $\beta > N$

$$\begin{aligned} u(h\beta) &= \frac{(h\beta)^3}{12} \sum_{\gamma=0}^N K_1[\gamma] - \frac{(h\beta)^2}{4} \sum_{\gamma=0}^N K_1[\gamma](h\gamma) \\ &\quad + \frac{(h\beta)}{4} \sum_{\gamma=0}^N K_1[\gamma](h\gamma)^2 - \frac{1}{12} \sum_{\gamma=0}^N K_1[\gamma](h\gamma)^3 + P_1(h\beta) = Q_1^+(h\beta). \end{aligned} \quad (24)$$

From (23) and (24) we have the following

$$u(h\beta) = \begin{cases} Q_1^-(h\beta), & \beta < 0, \\ F_4(h\beta), & 0 \leq \beta \leq N, \\ Q_1^+(h\beta), & \beta > N, \end{cases} \quad (25)$$

here

$$Q_1^-(h\beta) = d_0^-(h\beta) + d_1^-, \quad (26)$$

$$Q_1^+(h\beta) = d_0^+(h\beta) + d_1^+ \quad (27)$$

$d_0^-, d_1^-, d_0^+, d_1^+$ – unknown constants.

First, let's find the values $K_1[\beta]$ for $\beta = 1, 2, \dots, N-1$

$$\begin{aligned} K_1[\beta] &= hD_2(h\beta) * u(h\beta) = h \sum_{\gamma=-1}^N D_2(h\beta - h\gamma) u(h\gamma) \\ &= h \sum_{\gamma=-1}^{-1} D_2(h\beta - h\gamma) u(h\gamma) + \sum_{\gamma=0}^N D_2(h\beta - h\gamma) F_4(h\gamma) + \sum_{\gamma=N+1}^N D_2(h\beta - h\gamma) u(h\gamma) \\ &= hD_2(h\beta) * F_4(h\beta) + h \sum_{\gamma=1}^N D_2(h\beta + h\gamma) (Q_1^-(-h\gamma) - F_4(-h\gamma)) \\ &\quad + h \sum_{\gamma=1}^N D_2(h\beta - h(N+\gamma)) (Q_1^+(h(N+\gamma)) - F_4(h(N+\gamma))) \end{aligned}$$

$$\begin{aligned}
 &= h^5 (3 - 2\sqrt{3}) \sum_{\gamma=1}^{\beta-1} q^{\gamma} (Q_1^{-}(-h\gamma) - F_4(-h\gamma)) + q^{N-\beta-1} \sum_{\gamma=1}^{\beta-1} q^{\gamma} (Q_1^{+}(h(N+\gamma)) - F_4(h(N+\gamma))) \\
 &= h \ a q^{\beta} + b q^{N-\beta}, \quad (28)
 \end{aligned}$$

where

$$a = \sqrt{3} h^4 \sum_{\gamma=1}^{\beta-1} q^{\gamma} (Q_1^{-}(-h\gamma) - F_4(-h\gamma)),$$

$$b = \sqrt{3} h^4 \sum_{\gamma=1}^{\beta-1} q^{\gamma} (Q_1^{+}(h(N+\gamma)) - F_4(h(N+\gamma))).$$

This means that after these notations the coefficients $K_1[\beta]$, $\beta = \overline{1, N-1}$ will take the following form:

$$K_1[\beta] = h \ a q^{\beta} + b q^{N-\beta}, \quad \beta = \overline{1, N-1}. \quad (29)$$

Now we find a and b . To do this, we calculate on the left side of (15) the following sum

$$S(h\beta) = \sum_{\gamma=0}^N K_1[\gamma] \frac{|h\beta - h\gamma|^3}{12} = K_1[0] \frac{(h\beta)^3}{6} + S_1(h\beta) - S_2(h\beta), \quad (30)$$

$$S_1(h\beta) = \sum_{\gamma=1}^{\beta-1} K_1[\gamma] \frac{(h\beta - h\gamma)^3}{6},$$

$$S_2(h\beta) = \sum_{\gamma=0}^N K_1[\gamma] \frac{(h\beta - h\gamma)^3}{12}.$$

Then calculate $S_1(h\beta)$, for this we will use (29)

$$\begin{aligned}
 S_1(h\beta) &= \sum_{\gamma=1}^{\beta-1} K_1[\gamma] \frac{(h\beta - h\gamma)^3}{6} = \sum_{\gamma=1}^{\beta-1} h \ a q^{\gamma} + b q^{N-\gamma} \frac{(h\beta - h\gamma)^3}{6} \\
 &= \frac{h^4}{6} \sum_{\gamma=1}^{\beta-1} a q^{\beta} q^{-\gamma} \gamma^3 + b q^{N-\beta} \sum_{\gamma=1}^{\beta-1} q^{\gamma} \gamma^3.
 \end{aligned}$$

To calculate this amount we will use the following formula [20]

$$q^{\gamma} \gamma^k = \frac{1}{1-q} \sum_{i=0}^k \frac{q^i}{1-q^{-i}} \Delta^i 0^k - \frac{q^n}{1-q} \sum_{i=0}^k \frac{q^i}{1-q^{-i}} \Delta^i \gamma^k \Big|_{\gamma=n} \quad (31)$$

here $\Delta^i \gamma^k$ this i -finite order difference γ^k , $\Delta^i 0^k = \Delta^i \gamma^k \Big|_{\gamma=0}$ and $\Delta^i 0^k = \sum_{\ell=1}^i (-1)^{i-\ell} C_i^{\ell} \ell^k$.

$$\begin{aligned}
 S_1(h\beta) &= \frac{h^4}{6} a q^{\beta} \sum_{i=0}^3 \frac{1}{1-q^{-i}} \frac{q^{-i}}{1-q^{-i}} \Delta^i 0^3 - \frac{q^{-\beta}}{1-q^{-1}} \sum_{i=0}^3 \frac{q^{-i}}{1-q^{-i}} \Delta^i \beta^3 \\
 &+ \frac{h^4}{6} b q^{N-\beta} \sum_{i=0}^3 \frac{1}{1-q} \frac{q^i}{1-q} \Delta^i 0^3 - \frac{q^{\beta}}{1-q} \sum_{i=0}^3 \frac{q^i}{1-q} \Delta^i \beta^3 \quad (32)
 \end{aligned}$$

here $q - E_2(q) = q^2 + 4q + 1$ - since this is the root of the Euler- Ferabenius polynomial of the second order

$$\sum_{i=0}^3 \frac{q^{-i}}{1-q^{-i}} \Delta^i 0^3 = 0, \quad \sum_{i=0}^3 \frac{q^i}{1-q} \Delta^i 0^3 = 0.$$

To mean

$$S_1(h\beta) = \frac{h^4}{6} - \frac{aq}{q-1} \sum_{i=0}^3 \frac{1}{q-1} \Delta^i \beta^3 - \frac{bq^N}{1-q} \sum_{i=0}^3 \frac{q}{1-q} \Delta^i \beta^3. \quad (33)$$

Using the following formula [19] we simplify (33)

$$\Delta^\alpha x^\nu = \sum_{p=0}^{\nu} \sum_{p=0}^{\nu} \Delta^\alpha 0^p x^{\nu-p}. \quad (34)$$

$$S_1(h\beta) = - \sum_{j=0}^3 \frac{(h\beta)^{3-j}}{j!(3-j)!} h^{j+1} \frac{aq}{q-1} \sum_{i=0}^3 \frac{1}{q-1} \Delta^i 0^j - \frac{bq^N}{1-q} \sum_{i=0}^3 \frac{q}{1-q} \Delta^i 0^j. \quad (35)$$

Calculate $S_2(h\beta)$

$$\begin{aligned} S_2(h\beta) &= \sum_{\gamma=0}^N K_1[\gamma] \frac{(h\beta - h\gamma)^3}{12} = \sum_{j=0}^3 \frac{(h\beta)^{3-j} (-1)^j}{2j!(3-j)!} \sum_{\gamma=0}^N K_1[\gamma] (h\gamma)^j \\ &= \sum_{j=2}^3 \frac{(h\beta)^{3-j} (-1)^j}{2j!(3-j)!} \sum_{\gamma=0}^N K_1[\gamma] (h\gamma)^j = \frac{h\beta}{4} \sum_{\gamma=0}^N K_1[\gamma] (h\gamma)^2 - \frac{1}{12} \sum_{\gamma=0}^N K_1[\gamma] (h\gamma)^3. \end{aligned} \quad (36)$$

We substitute equalities (35) and (36) into (15), equate the corresponding powers $(h\beta)$

$$K_1[0] = h \frac{aq - bq^N}{q-1}, \quad (37)$$

$$\frac{aq + bq^{N+1}}{(q-1)^2} = \frac{B_4 h^2}{24}, \quad (38)$$

$$\begin{aligned} P_1(h\beta) &= (h\beta) \frac{h^3(aq - bq^N(1+q))}{12(1-q)} + \frac{1}{4} \sum_{\gamma=0}^N K_1[\gamma] (h\gamma)^2 - \frac{h^4}{1440} \\ &\quad - \frac{1}{12} \sum_{\gamma=0}^N K_1[\gamma] (h\gamma)^3 + \frac{h^4}{2880}. \end{aligned} \quad (39)$$

In equality (16) using (29) and (37) we find the form $K_1[N]$.

$$K_1[N] = h \frac{bq - aq^N}{q-1}. \quad (40)$$

As stated above, we can expand equality (17) using (34), (37) and (40)

$$\begin{aligned} \sum_{\gamma=0}^N K_1[\gamma] (h\gamma) &= h^2 \sum_{i=0}^1 \frac{aq^i + bq^{N+1}(-1)^{N+1}}{(1-q)^{i+1}} \Delta^i 0^1 \\ &\quad - \sum_{p=0}^1 \frac{h^{p+1}}{p!(1-p)!} \sum_{i=0}^1 \frac{aq^{N+i} + bq(-1)^{N+1}}{(1-q)^{i+1}} \Delta^i 0^p + h \frac{aq^N}{1-q} + h \frac{bq}{q-1} = 0. \end{aligned} \quad (41)$$

From equality (41) we equate the corresponding powers h

$$\frac{aq + bq^{N+1}}{(1-q)^2} = \frac{aq^{N+1} + bq}{(1-q)^2}, \quad (42)$$

and from (38) and (42) we find a and b

$$a = \frac{h^2}{120(1+q^N)}, \quad b = \frac{h^2}{120(1+q^N)}, \text{ i.e. } a = b.$$

Theorem 2 is completely proved.

NUMERICAL RESULTS

We present the numerical values of $\|\ell_N\|_{L_2^{(4)}}^*$ of the quadrature formula in $L_2^{(4)}(0,1)$. In addition, we will calculate integrals of specific functions. We define $|R_N(f)|$ as the absolute error of the optimal quadrature formula in space $L_2^{(4)}(0,1)$. Then from the Cauchy-Schwartz inequality we possess the following:

$$|R_N(f)| \leq \|f\|_{L_2^{(4)}} \|\ell_N\|_{L_2^{(4)}}^*$$

the norm of $\|\ell_N\|_{L_2^{(4)}}^*$ of the optimal quadrature formula of the Hermite type constructed in space $L_2^{(4)}(0,1)$ as follows.

$$f(x) = \ln(x+1) + x^7, \quad I = \int_0^1 f(x) dx.$$

Table 1. We present the values of the norm of the error functional (3) of the optimal quadrature formula (1) in space $L_2^{(4)}(0,1)$ at $N = 10$, $N = 100$ and $N = 1000$.

TABLE 1. Squared norm of error functional of optimal quadrature formula

	$N = 10$	$N = 100$	$N = 1000$
$\ \ell_N\ _{L_2^{(4)}}^*$	$9.38093 * 10^{-15}$	$8.37829 * 10^{-23}$	$8.27817 * 10^{-31}$

Values of the integral of the function $f(x)$ at $N = 10; 20; 30; 40; 50$ calculate using the optimal quadrature formula of the Hermite type in space $L_2^{(4)}(0,1)$ [14] and denote it as **O₁EM**

At the same time, the values of the integral of the function $f(x)$ at $N = 10; 20; 30; 40; 50$. Let us calculate using the coefficients of the optimal quadrature formula of the Euler-Maclaurin type constructed in space $L_2^{(4)}(0,1)$ and denote it as **O₂EM**.

TABLE 2. Error of optimal quadrature formula

N	I (Exact value)	 I-O₁EM 	 I- O₂EM
10	0.5112943610	$4.4 * 10^{-6}$	$3.2 * 10^{-6}$
20	0.5112943610	$1.4 * 10^{-7}$	$1.0 * 10^{-7}$
30	0.5112943610	$1.9 * 10^{-8}$	$1.3 * 10^{-8}$
40	0.5112943610	$4.5 * 10^{-9}$	$3.0 * 10^{-9}$
50	0.5112943610	$1.4 * 10^{-9}$	$9.0 * 10^{-10}$

CONCLUSION

The values of the function up to the second-order derivative at the nodal points are given. Using these values, we constructed a quadrature formula with a derivative to calculate the exact integral of the function. We found that the error functional corresponding to the difference between this quadrature sum and the definite integral appears. The error functional is a

multivariate function of the coefficients of the quadrature formula. We have constructed the Lagrange function to find the conditional extremums of a multivariable function. To find the minimum of this function in terms of coefficients, we got a system of linear algebraic equations depending on the coefficients of the quadrature formula. We used the discrete analogue d^4/dx^4 to solve this system using the Sobolev method. Solving the system, we found the analytical view of the coefficients. Using these coefficients, we calculated the norm of the error function. We analyzed our theoretical results in numerical experiments.

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