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CAUCHY PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION: THEORETICAL FOUNDATIONS AND ANALYTICAL TECHNIQUES

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Abstract: The Cauchy problem for ordinary differential equations (ODEs) plays a pivotal role in the theory and applications of differential equations. It involves solving a differential equation given an initial condition, often modeling time-dependent physical, biological, or economic systems. In this study, we rigorously examine the foundational aspects of the Cauchy problem, focusing on conditions for existence and uniqueness of solutions. Through theoretical results such as the Picard–Lindelöf theorem and Peano's existence theorem, we explore different solution behaviors. Furthermore, illustrative examples and computational approaches are provided to support the theoretical exposition. This work lays a solid groundwork for further studies in the theory of dynamical systems and numerical methods.

Key words: ordinary differential equations, Lipschitz condition, solution, Picard–Lindelöf theorem.

1. Introduction

The study of ordinary differential equations is central to the mathematical modeling of dynamic phenomena. A significant class of problems in this area involves the Cauchy initial value problem, where a solution is required to satisfy both a differential equation and an initial condition. This formulation is ubiquitous in real-world applications, from Newtonian mechanics and population dynamics to electrical circuits and control systems.

The general form of the Cauchy problem for a first-order ODE is:

$$\frac{dy}{dx} = f(x, y), y(x^0) = y^0$$

Solving this problem entails determining a function $y(x)$that satisfies the differential equation on some interval containing x^0, and that takes the prescribed value y^0at $x = x^0$.
The questions of existence, uniqueness, and continuity with respect to initial conditions are critical in both theoretical and applied contexts. These properties ensure that models behave predictably and that simulations are stable under perturbations.

2. Methods

2.1 Classical Theorems of Existence and Uniqueness

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2.1.1 Picard–Lindelöf Theorem (Local Existence and Uniqueness)
If f(x, y) is continuous in a neighborhood of (x_0, y_0) and satisfies a Lipschitz condition in y, then there exists a unique solution y(x) to the initial value problem in some interval $[x^0 - h, x^0 + h]$, where h > 0.

Lipschitz condition:
There exists a constant L > 0 such that for all x in an interval and all y_1, y_2,
$\left|f(x, y^1) - f(x, y^2)\right| \leq L\left|y^1 - y^2\right|$

2.1.2 Peano's Existence Theorem (Without Uniqueness)
If f(x, y)is merely continuous, then at least one solution exists on some interval around x^0, but uniqueness is not guaranteed.

2.2 Successive Approximations (Picard Iteration)

An effective method to construct the solution is the Picard iteration scheme:

$y^0(x) = y^0$
$y_n{}^{+1}(x) = y^0 + \int_{x^0}{}^{x} f\left(t, y_n(t)\right) dt$

This sequence converges uniformly to the actual solution under the assumptions of the Picard–Lindelöf theorem. The iteration provides both theoretical insight and a practical numerical method for approximating solutions.

3. Results

3.1 Example 1: Exponential Growth

$$\frac{dy}{dx} = ky, y(0) = y^0$$

Here, $f(x, y) = ky$ is linear in y, continuous, and satisfies the global Lipschitz condition. By direct integration:

$y(x) = y^0 e^{kx}$

The solution exists for all x $\in \mathbb{R}$, demonstrating global existence and uniqueness.

3.2 Example 2: Non-uniqueness (Violation of Lipschitz Condition)

$$\frac{dy}{dx} = \sqrt{|y|}, y(0) = 0$$

This function is continuous but not Lipschitz near y = 0. Multiple solutions exist, including:

$$y(x) = 0 \; and \; y(x) = \left(\frac{x}{2}\right)^2$$

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Thus, Peano's theorem applies, but uniqueness fails.

3.3 Numerical Approximation

Using Picard iteration for $f(x, y) = y$, with $y(0) = 1$:

$$y^0(x) = 1$$
$$y^1(x) = 1 + x$$
$$y^2(x) = 1 + x + \frac{x^2}{2}$$
$$y^3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

The sequence converges to y(x) = e^x, which is the exact solution.

4. Discussion

The results affirm that regularity conditions on f(x, y) critically determine the behavior of solutions to the Cauchy problem. The Picard–Lindelöf theorem provides both a constructive and theoretical framework for solution analysis. However, in real-world problems, discontinuities, singularities, and non-smooth behaviors frequently arise, which violate Lipschitz conditions. In such cases, generalizations such as Carathéodory's theorem or Filippov's theory for differential inclusions are necessary.

Furthermore, numerical methods like Euler's method, Runge–Kutta methods, and multi-step methods rely on the existence and uniqueness of solutions to ensure convergence and stability. Understanding the conditions under which solutions exist helps guide both analytical and computational approaches.

5. Conclusion

The Cauchy problem encapsulates the core challenge in solving ordinary differential equations—predicting the evolution of systems from known initial conditions. This paper has presented a comprehensive treatment of the problem, emphasizing existence and uniqueness results, illustrated with classical examples, and extended to computational techniques. These results not only reinforce the theoretical structure of ODEs but also serve as a launching point for more advanced study in applied mathematics, dynamical systems, and numerical analysis.

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