JOURNAL OF APPLIED SCIENCE AND SOCIAL SCIENCE
eISSN 2229-3113 pISSN 2229-3205
Volume 14 Issue 06, June 2024
Impact factor: 2019: 4.679 2020: 5.015 2021: 5.436, 2022: 5.242, 2023: 6.995, 2024 7.75

ON THE FORMULATION AND STUDY OF A BOUNDARY VALUE PROBLEM FOR A FOURTH-ORDER EQUATION OF PARABOLIC-HYPERBOLIC TYPE IN A PENTAGONAL DOMAIN

Mamajonov Sanjarbek Mirzaevich

Associate Professor of Kokand University

e-mail: sanjarbekmamajonov@gmail.com

Abstract. In this paper, one boundary value problem is set and investigated for a fourth-order parabolic-hyperbolic equation. The problem is solved by the method of directly constructing a solution to the problem.

Keywords: fourth-order equation, parabolic-hyperbolic type, boundary value problem, Green's function of the first and second boundary value problems, Volterra integral equation of the second kind.

1. Introduction

Fundamental research on mixed second-order equations of elliptic-hyperbolic type began in the 1920s by the Italian mathematician Tricomi and was developed by Gellerstedt, A.V.Bitsadze, K.I.Babenko, I.L.Karol, F.I.Frankl, M.M. Smirnov, M.S. Salakhitdinov, T.D. Dzhuraev and others.

Research into equations of elliptic-parabolic and parabolic-hyperbolic types began in the 50-60s of the last century.

Then, in the 70-80s of the twentieth century, various problems for equations of the third and high orders of parabolic-hyperbolic type began to be studied. Such problems were studied mainly by T.D. Dzhuraev and his students (for example, see [1,2]).

Currently, the study of boundary value problems for third- and high-order equations of parabolic-hyperbolic type has been developing in a broad sense (for example, see [3], [4]).

The study of many problems in gas dynamics, the theory of elasticity, the theory of plates and shells is led to consideration in high-order partial differential equations. From a physical point of view, fourth-order differential equations are also of great interest (see [5]-[10]).

JOURNAL OF APPLIED SCIENCE AND SOCIAL SCIENCE
eISSN 2229-3113 pISSN 2229-3205
Volume 14 Issue 06, June 2024
Impact factor: 2019: 4.679 2020: 5.015 2021: 5.436, 2022: 5.242, 2023: 6.995, 2024 7.75

2. Statement of the problem

In this work, in the pentagonal region G of the plane xOy, one boundary value problem for a fourth-order parabolic-hyperbolic equation of the form

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)(Lu)=0, \qquad (1)$$

is posed and studied. Where $G=G_1 \cup G_2 \cup G_3 \cup J_1 \cup J_2$; $G_1 -$ square with vertices at points $A(0;0)$, $B(1;0)$, $B_0(1,1)$, $A_0(0,1)$; $G_2 -$ triangle with vertices at points B, $C(0,-1)$, $D(-1,0)$; $G_3 -$ square with vertices at points A, D, $D_0(-1,1)$, A_0; $J_1 -$ open segment with vertices at points B, D; $J_2 -$ open segment with vertices at points A, A_0;

$$Lu=\begin{cases} u_{xx}-u_y, \ (x,y)\in D_1, \\ u_{xx}-u_{yy}, \ (x,y)\in D_i, \ i=2,3. \end{cases}$$

For equation (1), the following problem is posed:

Problem A**.** Find function $u(x,y)$, that

1) is continuous in $\overline{G}$ and in the region $G \setminus J_1 \setminus J_2$ has continuous derivatives participating in equation (1), and u_x, u_y, u_{xx}, u_{xy}, u_{yy} are continuous in G up to part of the boundary of the region G, indicated in the boundary conditions;

2) satisfies equation (1) in the area $G \setminus J_1 \setminus J_2$;

3) satisfies the following boundary conditions:

$$u(1,y)=\varphi_1(y), \ 0\le y\le 1; \qquad (2)$$

$$u(-1,y)=\varphi_2(y), \ 0\le y\le 1; \qquad (3)$$

$$u_x(-1,y)=\varphi_3(y), \ 0\le y\le 1; \qquad (4)$$

$$u|_{CF}=\psi_1(x), \ 0\le x\le \frac{1}{2}; \qquad (5)$$

JOURNAL OF APPLIED SCIENCE AND SOCIAL SCIENCE
eISSN 2229-3113 pISSN 2229-3205
Volume 14 Issue 06, June 2024
Impact factor: 2019: 4.679 2020: 5.015 2021: 5.436, 2022: 5.242, 2023: 6.995, 2024 7.75

$$u\big|_{CD} = \psi_2(x), \ -1 \quad x \quad 0; \tag{6}$$

$$\left.\frac{\partial u}{\partial n}\right|_{BC} = \psi_3(x), \ 0 \quad x \quad 1; \tag{7}$$

$$\left.\frac{\partial u}{\partial n}\right|_{CD} = \psi_4(x), \ -1 \quad x \quad 0; \tag{8}$$

$$\left.\frac{\partial^2 u}{\partial n^2}\right|_{CD} = \psi_5(x), \ -1 \quad x \quad 0; \tag{9}$$

4) satisfies the following gluing conditions on type change lines:

$$u(x,+0) = u(x,-0) = T(x), \ -1 \quad x \quad 1; \tag{10}$$

$$u_y(x,+0) = u_y(x,-0) = N(x), \ -1 \quad x \quad 1; \tag{11}$$

$$u_{yy}(x,+0) = u_{yy}(x,-0) = M(x), \ -1 < x < 1; \tag{12}$$

$$u_{yyy}(x,+0) = u_{yyy}(x,-0) = \Theta(x), \ -1 < x < 1; \tag{13}$$

$$u(+0,y) = u(-0,y) = \tau_3(y), \ 0 \quad y \quad 1; \tag{14}$$

$$u_x(+0,y) = u_x(-0,y) = \nu_3(y), \ 0 \quad y \quad 1; \tag{15}$$

$$u_{xx}(+0,y) = u_{xx}(-0,y) = \mu_3(y), \ 0 < y < 1, \tag{16}$$

where $\varphi_i \left(i = \overline{1,3}\right)$, $\psi_j \left(j = \overline{1,5}\right)$ – given sufficiently smooth functions, $n-$ internal normal to line $x + y = -1$ or $x - y = 1$, and $F(1/2,-1/2)$,

$$T(x) = \begin{matrix} \tau_1(x), \ if \ 0 \quad x \quad 1, \\ \tau_2(x), \ if \ -1 \quad x \quad 0; \end{matrix}$$

$$N(x) = \begin{matrix} \nu_1(x), \ if \ 0 \quad x \quad 1, \\ \nu_2(x), \ if \ -1 \quad x \quad 0; \end{matrix}$$

JOURNAL OF APPLIED SCIENCE AND SOCIAL SCIENCE
eISSN 2229-3113 pISSN 2229-3205
Volume 14 Issue 06, June 2024
Impact factor: 2019: 4.679 2020: 5.015 2021: 5.436, 2022: 5.242, 2023: 6.995, 2024 7.75

$$M(x)= \begin{matrix} \mu_1(x),\ if\ \ 0<x<1, \\ \mu_2(x),\ if\ \ -1<x<0; \end{matrix}$$

$$\Theta(x)= \begin{matrix} \theta_1(x),\ if\ \ 0<x<1, \\ \theta_2(x),\ if\ -1<x<0; \end{matrix}$$

$\tau_i, \nu_i, \mu_i\ (i=\overline{1,3})$, θ_1,θ_2 are unknown yet sufficiently smooth functions.

3. Investigation of the problem

Theorem. If $\varphi_1,\varphi_2\ \ C^4[0,1]$, $\varphi_3\ \ C^3[0,1]$, $\psi_1\ \ C^4[0,1/2]$, $\psi_2\ \ C^4[-1,0]$, $\psi_3\ \ C^3[0,1]$, $\psi_4\ \ C^3[-1,0]$, $\psi_5\ \ C^2[-1,0]$ and the matching conditions $\psi_1(0)=\psi_2(0)$, $\varphi_2(0)=\psi_2(-1)$, $\psi_4(0)=-\psi_3(0)$ are satisfied, then problem *A* has a unique solution.

Here we give an idea of the proof of this theorem. We prove the theorem by constructing a solution. We rewrite equation (1) in the form

$$u_{1xx}-u_{1y}=\omega_{11}(x-y)+\omega_{12}(x),\ (x,y)\ \ G_1, \qquad (17)$$

$$u_{jxx}-u_{jyy}=\omega_{j1}(x-y)+\omega_{j2}(x),\ (x,y)\ \ G_j\ \ (j=2,3), \qquad (18)$$

where the notation $u(x,y)=u_j(x,y),\ (x,y)\ \ G_j\ \left(j=\overline{1,3}\right)$ is introduced, and the functions $\omega_{j1}(x-y),\omega_{j2}(y)\ \left(j=\overline{1,3}\right)$ are still unknown and sufficiently smooth functions to be determined.

We will carry out the study first in the region G_2. Writing the solution to equation (18) $(i=2)$, satisfying conditions (10), (11) at $0\ \ x\ \ 1$ and substituting this solution into conditions (7), (8) and (9) after some calculations and transformations, we find function $\omega_{21}(x-y)+\omega_{22}(x)$.

Now substituting this solution into (6), we have the first relation between the unknown functions $T(x)$ and $N(x)$ in the interval $-1\ \ x\ \ 1$.

JOURNAL OF APPLIED SCIENCE AND SOCIAL SCIENCE
eISSN 2229-3113 pISSN 2229-3205
Volume 14 Issue 06, June 2024
Impact factor: 2019: 4.679 2020: 5.015 2021: 5.436, 2022: 5.242, 2023: 6.995, 2024 7.75

a) This relation at $-1 \quad x \quad 0$ has the form

$$\tau_2(x)-\nu_2(x)=\delta_1(x),-1 \quad x \quad 0. \qquad (19)$$

Next, substituting the solution that we wrote down above into (5), we obtain the second relation

$$\tau_2(x)+\nu_2(x)=\alpha_1(x),-1 \quad x \quad 0, \qquad (20)$$

where $\delta_1(x)$ and $\alpha_1(x)$ are known functions.

From (19) and (20) we find the functions $\tau_2(x)$, $\tau_2(x)$, $\nu_2(x)$.

b) At $0 \quad x \quad 1$, the relationship between the unknown functions $T(x)$ and $N(x)$ has the form

$$\tau_1(x)-\nu_1(x)=\alpha_1(x), 0 \quad x \quad 1. \qquad (21)$$

Now passing in equation (18) $(i=2)$ to the limit at $y \to 0$, by virtue of (10) and (12) at $0 \quad x \quad 1$ we obtain the relation:

$$\mu_1(x)=\tau_1(x)-\omega_{21}(x)-\omega_{22}(x). \qquad (22)$$

Differentiating equation (18) $(i=2)$ by y and putting $y \to 0$ in the resulting equation, we have

$$\nu_1(x)-\theta_1(x)=-\omega_{21}(x), 0 \quad x \quad 1. \qquad (23)$$

Next, applying the operator $\frac{\partial}{\partial y} \ \frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ to equation (17) and directing y to zero, we obtain another relation:

$$\nu_1(x)-\mu_1(x)+\mu_1(x)-\theta_1(x)=0.$$

Eliminating the functions $\nu_1(x)$, $\mu_1(x)$ and $\theta_1(x)$ from (21), (22), (23) and the last equation, then integrating the resulting equation three times from 0 to x, we arrive at a first-order linear ordinary differential equation for $\tau_1(x)$. Solving this equation for the known four conditions, we find function $\tau_1(x)$.

JOURNAL OF APPLIED SCIENCE AND SOCIAL SCIENCE
eISSN 2229-3113 pISSN 2229-3205
Volume 14 Issue 06, June 2024
Impact factor: 2019: 4.679 2020: 5.015 2021: 5.436, 2022: 5.242, 2023: 6.995, 2024 7.75

Then functions $\nu_1(x)$, $\mu_1(x)$, $u_2(x,y)$ will also be known.

Passing in equations (18) $(i=2)$ and (18) $(i=3)$ to the limit at $y\to 0$ taking into account conditions (10), (12), we find

$$\omega_{31}(x)+\omega_{32}(x)=\omega_{21}(x)+\omega_{22}(x), -1 \quad x \quad 0.$$

Next, differentiating equations (18) $(i=2)$ and (18) $(i=3)$ by y and in the resulting equations passing to the limit at $y\to 0$ taking into account conditions (11), (13), after some transformations we obtain

$$\omega_{31}(x-y)+\omega_{32}(x)=\omega_{21}(x-y)+\omega_{22}(x), -1 \quad x-y \quad 0.$$

Moving on to consider the problem in the region G_3, using the continuation method we obtain the first relation between the unknown functions $\tau_3(y)$ and $\nu_3(y)$:

$$\nu_3(y)=\tau_3(y)+\beta_1(y), 0 \quad y \quad 1. \qquad (24)$$

Next, passing in equations (17) and (18) $(i=3)$ to the limit at $x\to 0$ due to conditions (13) and (15), we find

$$\mu_3(y)-\tau_3(y)=\overline{\omega}_{11}(-y)+\omega_{12}(0), \qquad (25)$$

$$\mu_3(y)-\tau_3(y)=\omega_{31}(-y)+\omega_{32}(0), \qquad (26)$$

where it should be

$$\omega_{11}(x-y)=\begin{matrix} \overline{\overline{\omega}}_{11}(x-y), 0 \quad x-y \quad 1, \\ \overline{\omega}_{11}(x-y), -1 \quad x-y \quad 0, \end{matrix}$$

and $\overline{\omega}_{11}(0)=\overline{\overline{\omega}}_{11}(0)$.

In equation (17), passing to the limit at $y\to 0$, we obtain

$$\overline{\overline{\omega}}_{11}(x)+\omega_{12}(x)=\tau_1(x)-\nu_1(x). \qquad (27)$$

JOURNAL OF APPLIED SCIENCE AND SOCIAL SCIENCE
eISSN 2229-3113 pISSN 2229-3205
Volume 14 Issue 06, June 2024
Impact factor: 2019: 4.679 2020: 5.015 2021: 5.436, 2022: 5.242, 2023: 6.995, 2024 7.75

Differentiating equation (17) by y and putting $y \to 0$ in the resulting equation due to conditions (11), (12), after some calculations we find the function $\overline{\overline{\omega}}_{11}(x)$. Substituting $\overline{\overline{\omega}}_{11}(x)$ in (27), we find $\omega_{12}(x)$. Thus, we find the function $\overline{\overline{\omega}}_{11}(x-y)+\omega_{12}(x)$.

Excluding the function $\mu_3(y)$ from (25), (26), after some calculations, taking into account $\omega_{32}(0)=0$, after some transformations we obtain the relation

$$\overline{\omega}_{11}(x-y)+\omega_{12}(x)= \tau_3(y-x)-\tau_3(y-x) +\gamma_3(x,y), \qquad (28)$$

where $\gamma_3(x,y)$ is a known function.

Now we move to the region G_1. Writing the solution to equation (17), satisfying conditions (2), (10) at $0 \quad x \quad 1$ and (14) and differentiating this solution with respect to x and setting $x \to 0$, taking into account equalities (24), (27), (28), after some calculations, we find $\tau_3(y)$, and thus the functions $\tau_3(y)$, $\nu_3(y)$, $\overline{\omega}_{11}(x-y)+\omega_{12}(y)$, $T_2(x)$. Then the functions $u_3(x,y)$ and $u_1(x,y)$ will be known. So, we have found the solution to problem *A* in a unique way.

Literature

1. Djurayev T.D. Krayeviye zadachi dlya uravneniy smeshannogo i smeshanno-sostavnogo tipov. Tashkent: Fan, 1979, 240 s. [in Russian].

2. Djurayev T.D., Sopuev A., Mamajanov M. Krayeviye zadachi dlya uravneniy parabolo-giperbolicheskogo tipa. Tashkent, Fan, 1986, 220 s. [in Russian].

3. Djurayev T.D., Mamajanov M. Krayeviye zadachi dlya odnogo klassa uravneniy chetvertogo poryadka smeshannogo tipa. Differensialniye uravneniya, 1986, t.22, №1, s.25-31. [in Russian].

4. Taxirov J.O. Krayeviye zadachi dlya smeshannogo parabolo-giperbolicheskogo uravneniya s izvestnoy i neizvestnoy liniyami razdela. Avtoreferat kandidatskoy dissertatsii. Tashkent, 1988. [in Russian].

5. Apakov Y.P., Mamajonov S.M. Boundary-value problem for the fourth-order equation with multiple characteristics in a rectangular domain. Journal of Mathematical Sciences. 2023. Vol. 272, №2. pp. 185-201.

6. Mamajonov S.M. The third boundary problem for a fourth-order nonhomogeneous equation with constant coefficients. Bull. Inst. Math. 2022. Vol. 5, №6. pp. 100-109.

JOURNAL OF APPLIED SCIENCE AND SOCIAL SCIENCE
eISSN 2229-3113 pISSN 2229-3205
Volume 14 Issue 06, June 2024
Impact factor: 2019: 4.679 2020: 5.015 2021: 5.436, 2022: 5.242, 2023: 6.995, 2024 7.75

7. Apakov Y.P., Mamajonov S.M. Krayevaya zadacha dlya neodnorodnogo uravneniya chetvertogo poryadka s postoyannimi koeffitsiyentami. Chelyabinskiy fiziko-matematicheskiy jurnal. 2023. T. 8, vip. 2. s. 157–172. [in Russian].

8. Apakov Y.P., Mamajonov S.M. Krayevaya zadacha dlya neodnorodnogo uravneniya chetvertogo poryadka s peremennimi koeffitsiyentami. Dokladi Akademii nauk Respubliki Uzbekistan. 2022. №4. C. 7-13. [in Russian].

9. Apakov Y.P., Mamajonov S.M. Krayevaya zadacha dlya neodnorodnogo uravneniya chetvertogo poryadka s mladshimi chlenami. Differensialniye uravneniy. 2023. T. 59, №2. C. 183-192. [in Russian].

10. Apakov Y.P., Mamajonov S.M. Resheniye vtoroy krayevoy zadachi dlya neodnorodnogo uravneniya chetvertogo poryadka s kratnimi xarakteristikami. Nauchniy vestnik NamGU. 2023. № 7. C. 70–80. [in Russian].