JOURNAL OF APPLIED SCIENCE AND SOCIAL SCIENCE
eISSN 2229-3113 pISSN 2229-3205
Volume 15 Issue 10, October 2025
Impact factor: 2019: 4.679 2020: 5.015 2021: 5.436, 2022: 5.242, 2023: 6.995, 2024 7.75

BESSEL EQUATION

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Abstract: This paper examines the nonlinear oscillations of a rigid body mounted on viscoelastic supports. The equations of motion of the system are derived from Lagrange's equations of the second kind for systems with a finite number of degrees of freedom. A solution method for the problem is developed, numerical results are obtained, and the influence of nonlinearity on displacement amplitudes is evaluated.

Keywords: nonlinear oscillations, rigid body, vibration, Lagrange equations, viscoelastic support, degree of freedom

1. Introduction

This

$$x^2 y'' + xy' + (x^2 - p^2)y = 0 \qquad (p = \text{const}) \quad (1)$$

differential equation in the form *Bessel's equation* is called.

The solution to this equation should be sought not in the form of a power series, as is the case for solving some equations with variable coefficients, but in the form of a product of a power series with some power of x:

$$y = x^r \sum_{k=0} a_k x^k . \qquad (2)$$

r Since the indicator is not exact, we can assume that the a_0 coefficient is non-zero. (2) expression

$$y = \sum_{k=0} a_k x^{k+r} .$$

we write in the form and find its derivatives:

$$y' = \sum_{k=0} (r + x) a_k x^{k+r-1}, \quad y'' = \sum_{k=0} (r + x)(r + x - 1) a_k x^{k+r-2}$$

We put these expressions into equation (1):

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$$x^2 \sum_{k=0} (r+x)(r+x-1)a_k x^{k+r-2} + x \sum_{k=0} (r+x)a_k x^{k+r-1} +$$

$$+ (x^2 - p^2) \sum_{k=0} a_k x^{k+r} = 0.$$

$r, r+1, r+2, \ldots, r+k$ By setting the coefficients in front of the x-values equal to zero, we obtain the following system of equations:

$$[r(r-1)+r-p^2]a_0 = 0 \text{ ёки } [r^2-p^2]a_0 = 0,$$

$$[(r+1)r+(r+1)-p^2]a_1 = 0 \text{ ёки } [(r+1)^2-p^2]a_1 = 0,$$

$$[(r+2)(r+1)+(r+2)-p^2]a_2 + a_0 = 0 \text{ ёки } [(r+2)^2-p^2]a_2 + a_0 = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[(r+k)(r+k+1)+(r+k)-p^2]a_k + a_{k-2} = 0 \text{ ёки } [(r+k)^2-p^2]a_k + a_{k-2} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(3)

Let's look at this equation:

$$[(r+k)^2-p^2]a_k + a_{k-2} = 0,$$

It can be written as follows:

$$r, r+1, r+2, \ldots, r+k$$

By agreement a_0 0 so,

$$r^2 - p^2 = 0$$

therefore $r_1 = p$ or $r_2 = p$.

First, we consider the solution in the case of $r_1 = p > 0$.

(3) From the system of equations, one by one, all the coefficients $a_1, a_2, \ldots$ are determined; a_0 remains optional.

For example, $a_0 = 1$ Let it be. In that case

$$a_k = -\frac{a_{k-2}}{k(2p+k)}.$$

By giving different values to к, we find the following:

If we put the found coefficients in the formula (2):

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$$y_1 = x^p \begin{array}{l} 1 - \dfrac{x^2}{2(2p+2)} + \dfrac{x^4}{2\ \ 4(2p+2)(2p+4)} - \\ - \dfrac{x^4}{2\ \ 4\ \ 6(2p+2)(2p+4)(2p+6)} + ... \end{array} . \qquad (5)$$

For any k, the coefficient of a_k *in equation (3) is*

$$(r_1 + k)^2 - p^2$$

Since all $a_{2\nu}$ coefficients are different from zero, they are determined.

Thus, the function y_1 is a particular solution of equation (1). Now we find a condition such that all $r_2 = -p$ coefficients are determined even when this condition and the second root a_k are given. This means that for any positive even integer *k*

$$(r_2 + k)^2 - p^2 \quad 0 \qquad (6)$$

or

$$r_2 + k \quad p$$

It only happens when violence is committed.

But $p = r_1$ means,

$$r_2 + k \quad r_1$$

Thus, condition (6) holds in this case

$$r_1 - r_2 \quad k$$

is equivalent to the inequality, where k is a positive even integer.

But,

$$r_1 = p, \quad r_2 = -p,$$

so,

$$r_1 - r_2 = 2p.$$

Thus, if *p* is not integer, we can write the second particular solution obtained by replacing *p* in expression (5) with —*p*:

$$y_2 = x^{-p} \begin{array}{l} 1 - \dfrac{x^2}{2(-2p+2)} + \dfrac{x^4}{2\ \ 4(-2p+2)(-2p+4)} - \\ - \dfrac{x^4}{2\ \ 4\ \ 6(-2p+2)(-2p+4)(-2p+6)} + ... \end{array} . \qquad (5')$$

The convergence of the series of degrees (5) and (5') for all values of x can be easily determined by the Dalembert sign. It is also known that the functions y_1 and y_2 are linearly arbitrary.

The multiplication of the solution y_1 by some constant is called the Besselnang function of all genus p-order and is denoted by the symbol Jn. The solution y_2 is denoted by the symbol J_{-p}.

Thus, when p is not an integer, the general solution of equation (1) is:

$$y = C_1 J_p + C_2 J_{-p}$$

For example, if $p = \frac{1}{2}$, then the series (5) takes the following form:

$$x^{\frac{1}{2}}\left|1 - \frac{x^2}{2(2p+2)} + \frac{x^4}{2\ \ 4(2p+2)(2p+4)} - ....\right| =$$

$$= \frac{1}{\sqrt{x}}\left|x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right|.$$

The multiplication of this solution by the constant factor $\sqrt{\frac{2}{\pi}}$ is called Bessel's $J_{1/2}$ function; we see that in the brackets there is a series whose sum is equal to the sinx function. So,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin\ x.$$

Similarly, if we use formula (5');

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos\ x.$$

The general integral of equation (l) is $p = \frac{1}{2}$, which is as follows:

$$y = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x).$$

Then, let p be an integer, denoted by $n(n \quad 0)$. In this case, the solution (5) is meaningful and is the first particular solution of equation (1). However, the solution (5') is meaningless because one of the multipliers in the denominator of the expansion becomes zero. When p = n is a positive integer, the Bessel function J_n (5) is defined by multiplying the series by the invariant multiplier $\frac{1}{2^n n!}$ (or by 1 if $p = 0$):

$$J_n(x) = \frac{x^n}{2^n n!}\left|\begin{matrix} 1 - \frac{x^2}{2(2p+2)} + \frac{x^4}{2\ \ 4(2p+2)(2p+4)} - \\ - \frac{x^4}{2\ \ 4\ \ 6(2p+2)(2p+4)(2p+6)} + \dots \end{matrix}\right|$$

$$J_n(x) = \sum_{\nu=0} \frac{(-1)^\nu}{\nu!(n+\nu)!} \left(\frac{x}{2}\right)^{n+2\nu} \qquad (7)$$

In this case, the second private solution

$$K_n(x) = J_n(x)\ln x + x^{-n} \sum_{k=0} b_k x^k$$

It can be shown that we need to search for the form.

We can determine the coefficients b_k by substituting this expression into equation (1).

The function obtained by multiplying the function $K_n(x)$, whose coefficients are found in this way, by some constant number is called the Bessel function of the second genus of the p-order.

This is the second solution of equation (1), which forms a linear arbitrary system with the first solution.

The general integral takes the form:

$$y = C_1 J_n(x) + C_2 K_n(x)$$

In this case

$$\lim_{x \to 0} K_n(x) =$$

We note that .

Therefore, if we want to consider the finite solution at $x=0$, we should take $C_2 = 0$ in formula (8).

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