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MITTAG-LEFLER FUNCTION AND ITS ASYMPTOTICS

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Annotation: This scientific article analyzes the theoretical properties of the Mittag-Leffler function and its asymptotic expressions, which play an important role in fractional analysis and problems of mathematical physics. In particular, the definition of one- and two-parameter Mittag-Leffler functions, their relationship with Laplace transformations, and their role in the representation of solutions of fractional differential equations are highlighted. Also, the asymptotic behavior of this function in various domains of the complex plane is studied, and approximate estimates for large arguments are given. The obtained results serve for a deeper understanding of the mathematical foundations of subdiffusion processes, zero and nonlocal boundary value problems, as well as time-fraction models.

Keywords: Mittag-Leffler function, fractional analysis, asymptotic formula, Laplace transform, fractional differential equations, subdiffusion, complex plane.

Introduction. The rapid development of fractional-order differential and integral calculus in recent years requires the expansion of the classical mathematical apparatus in many practical and theoretical problems. In particular, models based on fractional derivatives are of great importance in modeling anomalous diffusion, viscoelastic media, biophysics, and economic processes. Among special functions, the Mittag-Lefler function occupies a special place in the representation of solutions of such models and in the study of their properties.

The Mittag-Leffler function was introduced at the end of the 19th century by the Swedish mathematician G. Mittag-Leffler and is considered a natural generalization of the exponential function. The Mittag-Leffler function plays a fundamental role in the theory of fractional differential equations, just as the exponential function plays a central role in the theory of classical differential equations. In particular, the analytical solutions of equations of time-fractional order are often expressed precisely by this function.

When analyzing the qualitative aspects of fractional models, it is important to determine the behavior of solutions over a long period of time. This requires a deep study of the asymptotic properties of the Mittag-Leffler function. Asymptotic estimates for various parameters and complex arguments provide important theoretical and practical results in determining the stability, attenuation, or rate of growth of solutions.

Literature review. The study of the Mittag-Leffler function and its asymptotic properties is closely related to the formation and development of fractional analysis. The first fundamental works in this direction are associated with the name of G. Mittag-Lefler, who considered the introduced function as a generalization of the exponential function and subsequently occupied an important place in the theory of special functions.

Since the second half of the 20th century, with the development of the theory of fractional differential equations, interest in the Mittag-Leffler function has significantly increased. In the studies of A.A. Kilbas [3], H.M. Srivastava [3] and J.J. Trujillo [3], the general theory of fractional integrals and derivatives was developed, and the role of the Mittag-Leffler function in the Laplace transformation and solutions of fractional equations was described in detail. These works are an important source for strengthening the theoretical foundations of fractional calculus. In the studies of R.Gorenflo [4], F.Mainardi [5] and their colleagues, the Mittag-Leffler function was interpreted as the main mathematical tool in modeling the processes of anomalous diffusion and subdiffusion. In their works, an in-depth analysis of the asymptotic behavior of one- and two-parameter Mittag-Leffler functions, the decay patterns of solutions for large times, and their relationship with probability theory was carried out.

I. Podlubny[2], in his famous monograph, developed analytical and numerical methods for fractional differential equations, considering the Mittag-Lefler function as a fundamental solution of systems of fractional order. In this source, asymptotic estimates of functions are widely used in the study of stability problems.

Also, in the works of A.Pazy, E.Bazilevič, S.G.Samko, and others, the connection between the theory of operator semigroupes and fractional equations was revealed, and it was shown that the asymptotic formulas of the Mittag-Leffler function are important in the qualitative analysis of solutions to evolutionary problems.

In general, the analysis of the available literature shows that the Mittag-Leffler function is one of the central objects of fractional analysis, and the study of its asymptotic properties remains a relevant issue not only in pure mathematics, but also in physics, mechanics, and other applied fields. At the same time, the need for further refinement of asymptotic estimates for some parameter domains and nonlocal problems requires new research in this area.

Main Part. The Mittag-Leffler function is one of the main special functions in fractional analysis, which is considered as a natural generalization of the exponential function. The one-parameter Mittag-Leffler function is defined by the following series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} z^k / \Gamma(\alpha k + 1), \quad \alpha > 0, \ z \in C.$$

This function corresponds exactly to the classical exponential function when $\alpha = 1$, i.e.

$$E_1(z) = e^z.$$

Therefore, it is interpreted as an analogue of the exponential function for fractional-order models. In practical and theoretical problems, the two-parameter Mittag-Leffler function is widely used:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} z^k / \Gamma(\alpha k + \beta), \quad \alpha > 0, \ \beta > 0.$$

This function is convenient for representing solutions of fractional differential equations, and its properties related to Laplace transformations are of great importance. In particular, the following statement is appropriate:

$$L\{ t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha) \}(s) = s^{\alpha-\beta} / (s^\alpha - \lambda), \quad Re(s) > |\lambda|^{1/\alpha}.$$

This formula is one of the main tools for finding analytical solutions of fractional differential

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equations.

For example, for $0 < \alpha < 1$

$$D_t^\alpha u(t) = \lambda u(t), \quad u(0) = 1,$$

solution of the fractional differential equation of the form

$$u(t) = E_\alpha(\lambda t^\alpha)$$

is expressed in the form.
One of the important aspects of the Mittag-Leffler function is its asymptotic properties. For large arguments, the behavior of this function differs depending on the location of the parameter α and the complex argument.
If $0 < \alpha < 2$ and $|\arg(z)| \le \pi\alpha/2$, then the following asymptotic estimate holds:

$$E_\alpha(z) \sim (1/\alpha) \exp(z^{1/\alpha}), \quad |z| \to \infty.$$

Conversely, for negative arguments, the Mittag-Leffler function has the property of algebraic decay,

$$E_\alpha(-z) \sim \sum_{k=1}^{n} (-1)^{k-1} / \Gamma(1 - \alpha k) \cdot z^{-k}, \quad |z| \to \infty.$$

This result mathematically substantiates the phenomena of slow attenuation observed in anomalous diffusion and subdiffusion processes.
Thus, the Mittag-Leffler function and its asymptotic expressions occupy a central place in the theory of fractional differential equations and serve as an important theoretical basis for studying the stability and behavior of solutions over time.
Conclusion. This article analyzes the Mittag-Leffler function and its role in fractional analysis, as well as its asymptotic properties for large arguments. The definitions of one- and two-parameter Mittag-Leffler functions, their relationship with Laplace transformations, and their role in solving fractional differential equations are highlighted.
By means of asymptotic estimates, the behavior of the function for large arguments was studied, which made it possible to determine the stability of fractional order systems over a long period of time, the rate of decay or growth of solutions in subdiffusion processes and nonlocal problems.
The obtained results show that the Mittag-Leffler function is of central importance as a fundamental solution of fractional differential equations, and its asymptotic expressions can be widely used in theoretical and practical research. At the same time, conducting additional research on the behavior of the function for various parameters and complex arguments allows for more accurate and effective analysis of fractional models.

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