eISSN: 2229-3213 pISSN: 2229-3205
The Journal of Applied Science and Social Science
Volume 16,Issue 02 February, 2026

ABOUT THE APPLICATION OF THE STEIN–TIKHOMIROV METHOD IN THE THEORY OF BRANCHING RANDOM PROCESSES

Yorkulova Gulfira Ravshan kizi

Karshi State University, master's student

E-mail: yorqulovagulfira@gmail.com

Tukhtaev Erkin Egamberdiyevich

Karshi State University, teacher

Annotatsiya. Galton-Vatson tarmoqlanuvchi tasodifiy jarayonlari amaliy jihatdan ko'pgina tatbiqlarga ega. Bu jarayonlarni strukturaviy va asimptotik tuzilishini o'rganishda matematik apparatlardan hosil qiluvchi funksiyalar, xarakteristik funksiyalar, Laplas almashtirishlari keng qo'llaniladi. Ushbu ishda biz limit teoremalarni isbotlash uchun Charlz Steyn tomonidan taklif qilingan ma'lum bitta usulning tatbiqini ko'rsatamiz. Rozenblatning aralash shartini qanoatlantiruvchi statsionar miqdorlar uchun markaziy limit teoremada yaqinlashish tezligini o'rganib, C.Steyn mos taqsimot funksiyalar orasidagi farq uchun ma'lum differensial identifikatsiyadan foydalangan. Keyinchalik, uning bu usuli A. Tixomirov tomonidan xarakteristik funksiyalar nuqtai nazaridan o'zgartirildi. Hozirgi vaqtda bu usul Steyn-Tixomirov (S-T) usuli deb ataladi va limit teoremalar sohasida keng qo'llanilmoqda.

Kalit so'zlar: tasodifiy miqdor, taqsimot funksiya, hosil qiluvchi funksiya, xarakteristik funksiya, sekin o'zgaruvchi funksiya, limit teorema

Annotation. Galton-Watson branching random processes have many practical applications. When studying the structural and asymptotic structure of these processes, generating functions, characteristic functions, and Laplace substitutions from mathematical apparatus are widely used. In this paper we show the application of a certain method proposed by Charles Stein for proving limit theorems. Studying the speed of convergence in the central limit theorem for stationary quantities satisfying Rosenblat's mixed condition, C. Stein used a certain differential identity for the difference between the corresponding distribution functions. Later, this method of his was modified by A. Tikhomirov in terms of characteristic functions. Currently, this method is called the Stein-Tikhomirov (S-T) method and is widely used in the field of limit theorems.

Key words: random variable, distribution function, generating function, characteristic function, slowly varying function, limit theorem

Introduction: We consider the classical Galton–Watson branching random process (G–W). The general description of the standard G–W process is as follows. There exists a population of identical particles, each of which may produce particles of the same type. The collection of particles at the initial moment of time is called the zero generation. Their "offspring," produced according to a random reproduction law, form the first generation, and so on. Each particle evolves according to a random law independently of the past history of the other particles and independently of the number of particles existing at the given moment of time. Let Z_n denote the population size at time $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0\} \cup \{\mathbb{N} = 1, 2, \ldots\}$. , . Unless otherwise specified, we always assume that P Z_0=1=1

In fact, the Galton–Watson process Z_n, n ϵN_0 forms a time-homogeneous Markov chain with a state space consisting of nonnegative integers and with transition probabilities given by

$$P_{ij} := \mathsf{P} \;\; Z_{n+1} = j | Z_n = i \;\; = \sum_{k_1+k_2+\cdots+k_i=j} p_{k_1} \cdot p_{k_2} \cdot \cdots \cdot p_{k_i}, \qquad (1)$$

for any , $i \in \mathbb{N},\ j, n \in \mathbb{N}_0$, где $p_k = P_{1k}$ и $\sum_{k\in\mathbb{N}_0} p_k = 1$.
Conversely, any Markov chain satisfying property (1) represents a Galton–Watson process with offspring distribution p_k , kϵ N_0 . From the above, it follows that the process p_k is completely determined by the specification of the offspring distribution Z_n, n ϵN_0 , and the random variable Z_n can be represented as the following sum of a random number of random variables:

$$Z_{n+1} = \xi_{n1} + \xi_{n2} + \cdots + \xi_{nZ_n}, \quad n \in \mathbb{N}_0, \qquad (2)$$

where the random variables ξ_{nk} are independent and have a common distribution P ξ_{11}=k= p_k. They are interpreted as the numbers of offspring of the -th particle in the -th generation; see [7, pp. 1–2], [19, p. 19]. From this point on, we assume that $p_k \neq 1$ and $p_0 + p_1 < 0$.

In the study of the properties of the Galton–Watson process, an important tool is the probability generating function (PGF) and its iterations. Let

$$F(s) = \sum_{k\in\mathbb{N}_0} p_k s^k, \quad 0 \le s < 1.$$

It is almost obvious that if the mathematical expectation $\Sigma_{k\epsilon N}$kp_k is finite, then the quantity $A := F'(s \uparrow 1) = \mathsf{E}\xi_{11}$ represents the mean number of immediate offspring produced by a single particle in one generation of the Galton–Watson process under consideration. For any kϵ N_0 , we denote by

$$\mathsf{E}_i s^{Z_n} := \sum_{j\in\mathbb{N}_0} \mathsf{P}_i \;\; Z_n = j \;\; s^j = \left[F_n(s)\right]^i,$$

the transition probability from state iϵ N to state jϵN in n steps of our Galton–Watson Markov process. Using the Kolmogorov–Chapman equation, it is easy to verify that the probability generating function (PGF)

$$\mathsf{E}_i s^{Z_n} := \sum_{j\in\mathbb{N}_0} \mathsf{P}_i \;\; Z_n = j \;\; s^j = \left[F_n(s)\right]^i,$$

where the probability generating function $F_n(s) = \mathsf{E}_1 s^{Z_n}$ (PGF) is given by the -fold

eISSN: 2229-3213 pISSN: 2229-3205
The Journal of Applied Science and Social Science
Volume 16,Issue 02 February, 2026

iteration of F(s) , that is, the following relations hold:

$$F_{n+m}(s) = F_n(F_m(s)) = F_m(F_n(s)); \qquad (3)$$

see [5], [7].

Equality (3) makes it possible to compute the numerical characteristics of the random variable Z_n for any n and plays an important role in the study of the asymptotic behavior of the trajectories of the G–W process. By directly differentiating relation (3), we obtain the expectation E Z_n= A^n and the variance

$$DZ_n = \begin{cases} \dfrac{\sigma^2 A^{n-1} \ A^n - 1}{A-1}, & A \neq 1, \\ \sigma^2 n \qquad\qquad , & A = 1, \end{cases}$$

where $\sigma^2 = F''(1) + A - A^2$. The above expressions for EZ_n and ED_n make it possible to classify Galton–Watson processes into three types depending on thealue of the parameter A . A G–W process is called subcritical, critical, or supercritical if A<1, A=1, or A>1, respectivel. For this reason, the parameter is sometimes referred to as the control (or regulating) parameter of the process.

In this paper, we demonstrate the possibility of applying a well-known method for proving limit theorems, the idea of which was proposed by C. Stein [10] in 1972. While studying the rate of convergence in the central limit theorem for stationary random variables satisfying Rosenblatt's mixing condition, C. Stein employed a certain differential identity for the difference between the corresponding distribution functions. Subsequently, his method was modified by A. Tikhomirov [1] in terms of characteristic functions. At present, this approach is known as the Stein–Tikhomirov (S–T) method and has gained considerable popularity in the field of limit theorems.

Literature Review : A simplified version of the S–T method in the case of normal approximation was proposed by Sh. Formanov [2]. It later became clear that the ideas underlying the S–T method can also be applied to other types of convergence. For example, in papers [3] and [4], these ideas were successfully used in proving limit theorems in the theory of branching processes. In the same works, several well-known classical theorems were reproved using the above approach. In the case of exponential approximation in the theory of Galton–Watson processes and Markov branching processes, the ideas of the S–T method were applied in works [13]-[17].

Research Methodology The essence of the Stein–Tikhomirov method can be explained by the following consideration. Suppose that it is required to prove a limit theorem for a sequence of distribution functions

$$\lim_{n\to\infty} G_n(s) = G(s) \qquad (4)$$

in the sense of weak convergence. Introduce the corresponding characteristic functions

$$\psi_n(\theta) = \int_{\mathbb{R}} e^{i\theta s} dG_n(s) \quad \text{and} \quad \psi(\theta) = \int_{\mathbb{R}} e^{i\theta s} dG(s)$$

and let $\varphi(\theta)$ be a solution of a certain differential equation. If $\varphi_n(\theta)$ asymptotically satisfies this equation, then the statement of the limit theorem (4) holds.

In the case of exponential approximation, the idea of the S–T method is clarified in the following lemma. Following Formanov [2], we introduce the class of Laplace transforms (LT)

$$\Gamma := \left\{ \varphi(\theta) : |\, \varphi'(0) \,| = \frac{1}{\alpha}, \ \ \theta \in \mathbb{R}_+ \right\}.$$

Within the class G introduce the operator

$$\mathbb{T}\left[\varphi(\theta)\right] := \varphi'(\theta) + \frac{1}{\alpha}\varphi^2(\theta). \tag{5}$$

It is easy to see that, the operator T[*] is an annihilating operator for the Laplace transform $\varphi_\alpha(\theta) := \int_{\mathbb{R}_+} e^{-\theta x} d\Gamma_\alpha(x)$ of the exponential law $\Gamma_\alpha(x) = 1 - e^{-\alpha x}$ that is,

$$\mathbb{T}\left[\varphi_\alpha(\theta)\right] \equiv 0. \tag{6}$$

Lemma 1. *Let $P_n(x)$, $n \in N$ - a family of distribution functions and the corresponding Laplace transforms (LT)*

$$\varphi_n(\theta) = \int_{\mathbb{R}_+} e^{-\theta x} dP_n(x) \in \Gamma.$$

In order for, as $n \to \infty$, the convergence to hold

$$\sup_x \left| P_n(x) - \Gamma_\alpha(x) \right| \longrightarrow 0, \tag{7}$$

it is necessary and sufficient that for, as $n \to \infty$,

$$\sup_{\theta \le \Theta} \left| \mathbb{T} \ \varphi_n(\theta) \right| \longrightarrow 0, \tag{8}$$

for any $\Theta > 0$.

Proof. The argument for the necessity part of condition (8) is based on the properties of the

Laplace transform. Indeed, from (5) and (6) it follows that

$$\left|\mathbb{T}\left[\varphi_n(\theta)\right]\right| = \left|\mathbb{T}\left[\varphi_n(\theta)\right] - \mathbb{T}\left[\varphi_\alpha(\theta)\right]\right|$$
$$\leq \left|\varphi_n'(\theta) - \varphi_\alpha'(\theta)\right| + \frac{1}{\alpha}\left|\varphi_n^2(\theta) - \varphi_\alpha^2(\theta)\right|. \qquad (9)$$

Since both the distribution function and the LT are bounded, differentiation and integration by parts yield

$$\left|\varphi_n'(\theta) - \varphi_\alpha'(\theta)\right| = \left|\int_{\mathbb{R}_+} xe^{-\theta x} d\ P_n(x) - \Gamma_\alpha(x)\right|$$
$$= \left|\int_{\mathbb{R}_+} \left[P_n(x) - \Gamma_\alpha(x)\right](1-\theta x)e^{-\theta x}dx\right|$$
$$\leq C_1 \cdot \sup_x \left|P_n(x) - \Gamma_\alpha(x)\right|, \qquad (10)$$

where $C_1 = C_1(0)$ is a positive constant for all $\theta \geq \Theta$ and $\Theta > 0$. On the other hand,

$$\left|\varphi_n^2(\theta) - \varphi_\alpha^2(\theta)\right| \leq 2\left|\varphi_n(\theta) - \varphi_\alpha(\theta)\right|. \qquad (11)$$

By virtue of relations (9)–(11), from (10) we obtain (8).

To prove the sufficiency of condition (8), we consider equation (5) as a differential equation with the initial condition $\varphi(0)=1$, and after some transformations, we obtain

$$\left|\varphi_n(\theta) - \varphi_\alpha(\theta)\right| = \varphi_n(\theta)\varphi_\alpha(\theta)\int_0^\theta \frac{\mathbb{T}\left[\varphi_n(\theta)\right]}{\varphi_n^2(\theta)}d\theta. \qquad (12)$$

From this, we verify the validity of the inequality

$$\sup_{\theta\leq\Theta}\left|\varphi_n(\theta) - \varphi_\alpha(\theta)\right| \leq C_2 \cdot \Theta \cdot \sup_{\theta\leq\Theta}\left|\mathbb{T}\ \varphi_n(\theta)\ \right|,$$

for any $\theta \leq \Theta$, where $C_2 = C_2(0)$ is a positive constant. The last inequality, by virtue of the continuity theorem, proves the sufficiency of condition (8) for the convergence (7).

Lemma proved**.**

Results and Discussion. Let us consider a critical Galton–Watson process with the probability generating function (PGF) $F(s) = \sum_{k\in\mathbb{N}_0} p_k s^k$ of the offspring distribution for a single particle and with finite variance. Let $R_n(s) := 1 - F_n(s)$. From the classical theory of

eISSN: 2229-3213 pISSN: 2229-3205
The Journal of Applied Science and Social Science
Volume 16,Issue 02 February, 2026

Galton–Watson processes, it is known that if $2B := F''(1) < \infty$, , then in this case the following asymptotic expansion holds

$$R_n(s) = \frac{1-s}{(1-s)Bn+1} \quad 1+o(1) \tag{13}$$

as n→∞, for all $0 \le s < 1$; see, e.g., [7, p. 19].

The ideas of the S–T method can also be extended to the case where the probability generating function admits, as $0 \le s < 1$, the representation

$$F(s) = s + (1-s)^{1+\nu} \mathcal{L}\left(\frac{1}{1-s}\right) \qquad \Re_\nu$$

where $0 \le s < 1$ and $\Im_x$ is a slowly varying function at infinity. From the main lemma in [20], we obtain the following relation:

$$\mathsf{P}\{\mathcal{H} > n\} = R_n(0) = \frac{\mathcal{N}(n)}{(\nu n)^{1/\nu}}, \tag{14}$$

where the function $\aleph_x$ is slowly varying at infinity and satisfies, as n→∞ the condition

$$\mathcal{N}(n) \cdot \mathcal{L}^{1/\nu}\left(\frac{(\nu n)^{1/\nu}}{\mathcal{N}(n)}\right) \longrightarrow 1.$$

.

Let us present a differential analogue of the main lemma.

Lemma 2. *Suppose condition holds* $\Re_\nu$. *Then, as* $S\uparrow 1$, *the following relation is valid:*

$$R_n'(s) = -\left(\frac{R_n(s)}{1-s}\right)^{1+\nu} \frac{\mathcal{L} \;\; 1/R_n(s)}{\mathcal{L}_1 \;\; 1/(1-s)}, \tag{15}$$

Where as $\Im_1.(n)/\Im(n) \to 1$, $n \to \infty$.

Lemma 3. *If A=1 and* $2B := F'(1)$ *the second moment is finite, then the following asymptotic representation holds:*

$$R_n'(s) = \frac{\hbar(s)B}{s-F(s)} R_n^2(s) \quad 1+o(1) \;, \tag{16}$$

As $n \to \infty$, *where* $F(s) \le h''(s) \le 1$, *and admits representation (13).*

Below, we will see that the application of the S–T method in combination with the differential analogue of the Main Lemma for critical processes is successfully used in the proof of limit theorems.

Let us state the following "local" version of Lemma 2.

Lemma 4. *If A=1 and the second momenti* $F''(1)$ *is finite, then, as* $s - 1$ *.the following asymptotic representation holds:*

$$R_n'(s) \sim -\left(\frac{R_n(s)}{1-s}\right)^2, \quad n \to \infty. \tag{17}$$

By combining expansions (13) and (17), we see that, as s↑1, the following relation holds:

$$R_n'(s) \sim -\left(\mathcal{V}_n^{(i)}(s)\right)^2, \quad n \to \infty, \tag{18}$$

where

$$\mathcal{V}_n^{(i)}(s) = \sum_{j\in\mathbb{N}} P_{ij}(n) s^j \text{ и } P_{ij}(n) = \mathsf{P}_i\left(Z_n = j \mid \mathcal{H} > n\right).$$

Let us consider the conditional distribution

$$G_n(x) := \mathsf{P}_i\left(q_n Z_n < x \mid \mathcal{H} > n\right),$$

where q_n=P, H>n. We introduce the corresponding Laplace transform (LT)

$$\psi_n(\theta) := \int_{\mathbb{R}} e^{-\theta s} dG_n(s) = \mathcal{V}_n^{(i)}\left(\theta_n\right),$$

Here $\theta_n = \exp\{-\theta q_n\}$. From the definition $\mathcal{V}_n^{(i)}(s)$, and relations $\mathcal{V}_n^{(i)}(s) \sim 1 - \frac{R_n(s)}{R_n(0)} = \frac{P_{11}(n)}{R_n(0)} \cdot \mathcal{M}_n(s),\ n \to \infty$ and (18), it follows that

$$\psi_n'(\theta) \sim -\mathcal{V}_n^2\left(\theta_n\right) \sim -\psi_n^2(\theta). \tag{19}$$

as n→∞. Since the Laplace transform of the exponential law is a solution of the equation $\varphi'(\theta)+\varphi^2(\theta)$ with the initial condition $\varphi(0)=1$, it follows from Lemma 1 that the relation (19) is sufficient for the convergence of $G_n(x) \to 1-e^{-x}$ This statement is known as Yaglom's Theorem; see [6].

Let H=min n: $Z_n=0$. If condition $\mathfrak{R}_v$ is satisfied, the following theorem holds.

Theorem 1. *Let the probability generating function (PGF) of the number of offspring in a Galton–Watson process satisfy condition* $\Re_\nu$ *Then the conditional distribution*

$$G_n(x) = \mathsf{P}_i\left\{ q_n Z_n < x \middle| \mathcal{H} > n \right\}$$

converges weakly to a non-degenerate law with Laplace transform

$$\Psi(\theta) = 1 - \frac{\theta}{\left(1+\theta^\nu\right)^{1/\nu}}. \tag{20}$$

This theorem, in the case $F(1-s)-(1-s) \in \Re_n^{1+\nu}$, was established in [9]. The proof presented below is considerably simpler than the proof given by the author in [9].

Proof of Theorem 1. Consider the Laplace transform

$$\Psi_n(\theta) := \int_{\mathbb{R}_+} e^{-\theta s} dG_n(s) = 1 - \frac{1-F_n^i(\theta_n)}{1-F_n^i(0)}, \quad \theta \in \mathbb{R}_+, \tag{21}$$

As n→∞ , where $\theta_n = \exp\{-\theta q_n\}$. Since $1-F_n^i(s) \sim iR_n(s)$, we have

$$\Psi_n(\theta) \sim 1 - \frac{R_n(\theta_n)}{q_n} \tag{22}$$

and, by differentiating with respect to θ, from (21) we obtain

$$\Psi_n'(\theta) \sim \theta_n R_n'(\theta_n) \tag{23}$$

Here, we use the statement of Lemma 2. Since $1-\theta_n \sim \theta q_n$, it follows from (14) that $\mathcal{L}_1\left(1/R_n(\theta_n)\right) \sim \mathcal{L}\left(1/(1-\theta_n)\right)$ and as h θ_n→∞, n→∞.

Therefore,

$$R_n'(\theta_n) \sim -\left(\frac{R_n(\theta_n)}{1-\theta_n}\right)^{1+\nu}, \tag{24}$$

as n→∞ . Из соотношений (21)–(24) находим, что при n→∞

$$\Psi_n'(\theta) \sim -\left(\frac{R_n \theta_n}{\theta q_n}\right)^{1+\nu} = -\left(\frac{1-\Psi_n(\theta)}{\theta}\right)^{1+\nu}. \quad (25)$$

We find that the Laplace transform of the distribution $G_n(x)$ asymptotically satisfies the differential equation

$$\Psi'(\theta) = -\left(\frac{1-\Psi(\theta)}{\theta}\right)^{1+\nu},$$

the solution of which, as is easily seen, is

$$\Psi(\theta) = 1 - \frac{1}{C + \theta^{-\nu}}^{1/\nu}.$$

Here, by virtue of the property $\lim_{\theta\to\infty} \Psi(\theta)$, we find C=1. Then, according to the idea of the Stein–Tikhomirov method, relation (25) is sufficient for the convergence of $\Psi_n(x)$ to the expression of the form (20).

Theorem proved.

Conclusion and Recommendations. Note that this theorem generalizes

Yaglom's theorem. Indeed, if n=1 , then it is easy to see that $\mathfrak{L}(n) \to F''(1)/2$

as n→∞ , and in this case the right-hand side of the Laplace transform in (21) takes the form $1+\theta^{-1}$, which is known to correspond to the standard exponential law. This fact once again confirms that the application of the theory of slowly varying functions allows us to investigate more deeply the structure and limiting properties of branching process.

References:

1. Тихомирова А.Н. О скорости сходимости в центральной предельной теореме для слабо зависимых величин. *Теория вер. и прим.*, 25(7), 1980, 800–818.

2. Форманов Ш.К. Метод Стейна-Тихомирова и неклассическая центральная предельная теорема. *Мат. заметки*, 71(7), 2002, 604–610.

3. Форманов Ш.К., Имомов А.А. Условные предельные теоремы для ветвящихся процессов при условии не вырождения «в далеком будущем». *Тезисы докл. конф. по теор. вероят. и мат. стат.*, Ташкент, 2000, 10–12 Мая, 55–56.

4. Форманов Ш.К., Абу Альгасем Альсинуси. О применении метода Стейна к доказательству предельных теорем теории ветвящихся процессов. *Узбекский мат. журнал*,

2, 1994, 54–58.

5. Харрис Т. *Теория ветвящихся случайных процессов*. Москва, 1966.

6. Яглом А.М. Некоторые предельные теоремы теории ветвящихся процессов. *ДАН СССР*, 56(8), 1947, 795–798.

7. Athreya, K.B., Ney, P.E. *Branching Processes*. Dover, Mineola, NY, 2004.

8. Lamperti, J., Ney, P.E. Conditioned branching processes and their limiting diffusions. *Theory of Probability and its Applications*, 13, 1968, 126–137.

9. Seneta, E. The stationary distribution of a branching process allowing immigration: A remark on the critical case. *Journal of the Royal Statistical Society, Series B*, 30(4), 1968, 176–179.

10. Stein, Ch. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, II, 1972, 583–602.

11. Steffensen, J.F. Deux problèmes du calcul des probabilités. *Annales de l'Institut Henri Poincaré*, 3, 1932, 319–344.

12. Steffensen, J.F. Om standsynligheden for at afkommet uddør. *Matematisk Tidsskrift*, B1, 1930, 19–23.

13. Имомов А.А. Метод Стейна-Тихомирова в теории критических ветвящихся процессов Гальтона-Ватсона. *Матер. V Межд. Ферганской конф.*, 2005, Май, Фергана, 110–114.

14. Имомов А.А. Дифференциальный аналог основной леммы теории марковских ветвящихся процессов и его применения. *Матер. конф. Сиражддиновские чтения*, Ташкент, 2004, 47–54.

15. Imomov, A.A. Locally-differential analogue of the basic lemma of the Galton–Watson processes and the Q-processes. *Abstracts of the III International Conference PCI'2010*, Baku, Azerbaijan, 182–186.

16. Imomov, A.A., Azimov, D.J. On one criterion of convergence to the exponential law. *Abstracts of the III International Conference PCI'2010*, Baku, 180–182.

17. Imomov, A.A. A differential analog of the main lemma of the theory of Markov branching processes and its applications. *Ukrainian Mathematical Journal*, 57(5), 2005, Springer, 307–315.

18. Севастьянов Б.А. Ветвящиеся процессы. *Наука, Москва*, 1971.

19. Jagers, P., Sagitov, S. General branching processes in discrete time as random trees. *Bernoulli*, 14(7), 2008, 949–962.

20. Imomov, A.A., Tukhtaev, E.E. On the application of slowly varying functions with remainder in the theory of Galton–Watson branching processes. *Journal of Siberian Federal University. Mathematics & Physics*, 12(1), 2019, 51–57.