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PROVING ALGEBRAIC INEQUALITIES

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Abstract

The problems covered in this article not only teach students to prove algebraic inequalities in mathematics, but also increase their interest in mathematics and help develop logical thinking, creative reasoning, and independent decision-making skills in problem situations.

Keywords

algebraic inequality, non-standard problem, Cauchy inequality.

INTRODUCTION

This article presents several examples related to algebraic inequalities along with their complete solutions. In addition, several problems are provided for students to solve independently. Studying problems on proving algebraic inequalities teaches students not to complete test assignments using ready-made formulas, but to reason deeply and solve problems thoughtfully. These types of problems are non-standard problems that, in many cases, cannot be solved directly by traditional methods.

1. Cauchy-Bunyakovsky inequality (AM-GM).

$a, b \quad 0$ for: $(a + b)/2 \quad \sqrt{ab}$ equality $a = b$ at

General form: $(a_1 + a_2 + \dots + a_n)/n \quad \sqrt[n]{a_1 a_2 \dots a_n}$

2. Cauchy-Schwarz inequality.

$$(a_1 I + a_2 I + \dots + a_n I)(b_1 I + b_2 I + \dots + b_n I) \quad (a_1 b_1 + a_2 b_2 + \dots + a_n b_n) I$$

3. Chebyshev inequality.

$a_1 \quad a_2 \quad \dots \quad a_n$ and $b_1 \quad b_2 \quad \dots \quad b_n$ for:

$n(a_1 b_1 + \dots + a_n b_n) \quad (a_1 + \dots + a_n)(b_1 + \dots + b_n)$

4. Bernoulli inequality.

$x > -1$ and n for: $(1 + x)^n \quad 1 + nx$

5. Jensen's inequality:

$f(x)$ for a convex function: $f((x_1 + \dots + x_n)/n) \quad (f(x_1) + \dots + f(x_n))/n$

Below we present the proofs of several algebraic inequalities.

PROBLEM 1. $a, b > 0$ for $a + b \quad 2\sqrt{ab}$ prove that.

Solution:

$$\left(\sqrt{a} - \sqrt{b}\right)^2 \quad 0 \text{— the perfect square is always non-negative}$$

$$a - 2\sqrt{ab} + b \quad 0$$

$$a + b \quad 2\sqrt{ab}$$

Equality holds when $a = b$ is satisfied.

PROBLEM 2. $a, b, c > 0$ for $a^2 + b^2 + c^2 \quad ab + bc + ca$ prove that.

Solution:

We multiply the inequality by 2:

$$2a^2 + 2b^2 + 2c^2 \quad 2ab + 2bc + 2ca$$

$$\left(a^2 - 2ab + b^2\right) + \left(b^2 - 2bc + c^2\right) + \left(c^2 - 2ca + a^2\right) \quad 0$$

$$(a - b)^2 + (b - c)\mathrm{I} + (c - a)\mathrm{I} \quad 0$$

Equality holds when $a = b = c$ is satisfied.

PROBLEM 3. $a, b > 0$ for $\frac{a}{b} + \frac{b}{a} \quad 2$ prove that.

Solution. Method 1.

We apply the AM-GM inequality:

$$\frac{a}{b} + \frac{b}{a} \quad \sqrt{\frac{a}{b} \ \frac{b}{a}} = \sqrt{1} = 1$$

$$\frac{a}{b} + \frac{b}{a} \quad 2$$

Method 2. $\frac{a}{b} + \frac{b}{a} - 2 = \left(a^2 + b^2 - 2ab\right) / (ab) = (a - b)^2 / (ab) \quad 0$

Equality holds when $a = b$ is satisfied.

PROBLEM 4. $a + b + c = 1, \ a, b, c > 0$ for $ab + bc + ca \quad \frac{1}{3}$

prove that.

Solution:

$$(a + b + c)\mathrm{I} = a\mathrm{I} + b\mathrm{I} + c\mathrm{I} + 2(ab + bc + ca) = 1$$

$$a\mathrm{I} + b\mathrm{I} + c\mathrm{I} = 1 - 2(ab + bc + ca)$$

From Problem 2:

$$a\mathrm{I} + b\mathrm{I} + c\mathrm{I} \quad ab + bc + ca$$

$$1 - 2(ab + bc + ca) \quad ab + bc + ca$$

$$1 \quad 3(ab + bc + ca)$$

$$ab + bc + ca \quad \frac{1}{3}$$

Equality holds when $a = b = c = \frac{1}{3}$ is satisfied.

PROBLEM 5. for a, b, c > 0, $(a + b)(b + c)(c + a) \quad 8abc$ prove that.

Solution:

From AM-GM:

$$a + b \quad 2\sqrt{ab}$$

$$b + c \quad 2\sqrt{bc}$$

$$c + a \quad 2\sqrt{ca}$$

We multiply all three inequalities:

$$(a + b)(b + c)(c + a) \quad 8\sqrt{ab}\ \sqrt{bc}\ \sqrt{ca} = 8\sqrt{a^2b^2c^2} = 8abc \ ✓$$

Equality holds when is satisfied when a = b = c.

PROBLEM 6. $a,\ b,\ c > 0$ and $abc = 1$ for $a + b + c \quad 3$ prove that.

Solution:

From the AM-GM inequality:

$$(a + b + c)/3 \quad \sqrt[3]{abc} = \sqrt[3]{1} = 1$$

$$a + b + c \quad 3$$

Equality holds when $a = b = c = 1$ is satisfied.

PROBLEM 7. $a, b, c > 0$ for $a^3 + b^3 + c^3 \quad 3abc$ prove that.

Solution:

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$a + b + c > 0$ (since $a, b, c > 0$)

From Problem 2: $a^2 + b^2 + c\text{I} - ab - bc - ca \quad 0$

Therefore: $a^3 + b^3 + c^3 - 3abc \quad 0$

$a^3 + b^3 + c^3 \quad 3abc$

Equality holds when $a = b = c$ is satisfied.

PROBLEM 8. $x, y, z > 0$ and $x + y + z = 1$ for $(1-x)(1-y)(1-z) \quad 8xyz$ prove that.

Solution:

$x + y + z = 1$ since:

$1 - x = y + z$
$1 - y = x + z$
$1 - z = x + y$

Therefore: $(1-x)(1-y)(1-z) = (y+z)(x+z)(x+y)$

From Problem 5: $(y+z)(x+z)(x+y) \quad 8xyz$.

Equality holds when $x = y = z = \frac{1}{3}$ is satisfied.

PROBLEM 9. for a, b > 0, prove that $a^4 + b^4 \geq a^3b + ab^3$.

Solution:

$$a^4 + b^4 - a^3b - ab^3 = a^3(a-b) - b^3(a-b) =$$
$$= (a-b)(a^3 - b^3) = (a-b)^2(a^2 + ab + b^2)$$

$(a-b)^2 \quad 0$ and $a^2 + ab + b^2 > 0$ ($a, b > 0$ for)

Therefore: $a^4 + b^4 - a^3b - ab^3 \quad 0$

Equality holds when $a = b$ is satisfied.

PROBLEM 10. $a, b, c > 0$ for $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \quad \frac{9}{a+b+c}$ prove that.

Solution:

From the Cauchy-Schwarz inequality:

$$(a+b+c)\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \qquad \sqrt{a}\ \frac{1}{a}+\sqrt{b}\ \frac{1}{b}+\sqrt{c}\ \frac{1}{c}\ ^{2} = (1+1+1)^2 = 9$$

Therefore, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \quad \frac{9}{a+b+c}$.

Equality holds when $a=b=c$ is satisfied.

PROBLEM 11. $a, b, c > 0$ for $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \quad \frac{3}{2}$ prove that.

Solution. We add 1 to each fraction:

$$\frac{a}{b+c}+1=\frac{a+b+c}{b+c}$$

From this

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+3=(a+b+c)\cdot\ \frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}$$ we obtain the equality.

Using the same method as in Problem 10:

$$(b+c)+(c+a)+(a+b)\cdot\ \frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b} \qquad 9$$

$$2(a+b+c)\cdot\ 1/(b+c)+1/(c+a)+1/(a+b) \qquad 9$$

Therefore, $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+3 \quad 9/2$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \quad \frac{3}{2}$$

Equality holds when $a=b=c$ is satisfied.

PROBLEM 12. $a, b, c > 0$ and $a+b+c=3$ for $a^2+b^2+c^2 \quad 3$ prove that.

Solution: Method 1.

Cauchy-Schwarz (or Quadratic mean ≥ Arithmetic mean):

$$\frac{a^2+b^2+c^2}{3} \quad \frac{a+b+c}{3}^{2} = (3/3)^2 = 1$$

$a^2+b^2+c^2 \quad 3$.

Method 2.

$$(a + b + c)^2 = a^2+b^2+c^2+2(ab + bc + ca) = 9$$
$$a^2+b^2+c^2 = 9-2(ab+bc+ca)$$

From Problem 2: $a^2+b^2+c^2 \quad ab+bc+ca$

$$3(a^2+b^2+c^2) \quad a^2+b^2+c^2+2(ab+bc+ca) = 9$$

$a^2+b^2+c^2 \quad 3$ ✓

Equality holds when $a=b=c=1$ is satisfied.

PROBLEM 13. $a, b, c > 0$ and $abc = 1$ for $\frac{1}{ai(b+c)}+\frac{1}{bi(c+a)}+\frac{1}{ci(a+b)} \quad \frac{3}{2}$ prove that.

Solution:

$abc = 1$ using, we transform:

$$\frac{1}{ai(b+c)} = \frac{bc}{a^2(b+c)}$$ (since $abc = 1$ from $a = \frac{1}{bc}$)

$x = \frac{1}{a},\ y = \frac{1}{b},\ z = \frac{1}{c}$ if we let $xyz = 1$ and

The inequality takes the form $\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \quad \frac{3}{2}$ as follows

From Cauchy-Schwarz:

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \cdot (y+z)+(z+x)+(x+y) \quad (x+y+z)^2$$

The second factor of the left bracket $= 2(x+y+z)$

Therefore, $\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \quad \frac{x+y+z}{2}$

$xyz=1$ and from AM-GM: $x+y+z \quad 3$

Therefore, $\frac{x+y+z}{2} \quad \frac{3}{2}$.

Equality holds when $a=b=c=1$ is satisfied.

Exercises for independent practice

1. $a,\ b,\ c>0$ for $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \quad a+b+c$ prove that.

2. $a,\ b,\ c$ — for the sides of a triangle $a^2+b^2+c^2<2(ab+bc+ca)$ prove that.

3. $x,\ y,\ z>0$ and $xyz=1$ for $x^2+y^2+z^2 \quad x+y+z$ prove that.

4. $a,\ b,\ c>0$ for $(a+b+c)^3 \quad 27abc$ prove that.

5. $a,\ b,\ c\ >0$ and $a+b+c=1$ for $\sqrt{a}+\sqrt{b}+\sqrt{c} \quad \sqrt{3}$ prove that.

6. $a,\ b,\ c\ >0$ and $abc\ =1$ for $a^3(b+c)+b^3(c+a)+c^3(a+b) \quad 2(ab+bc+ca)$ prove that.

7. $a,\ b,\ c \quad 0$ for $a^4+b^4+c^4 \quad abc(a+b+c)$ prove that.

CONCLUSION

It is known that studying problems on proving inequalities is somewhat challenging for students. If students are taught more examples like the ones above and practice them repeatedly, they can acquire knowledge, skills, and competencies just like in other topics. The problems covered in this article also lay the groundwork for students to study olympiad problems.

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