ON THE APPLICATIONS OF REGULARLY VARYING FUNCTIONS IN THE THEORY OF BRANCHING PROCESSES

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Abstract: We consider a discrete-time Galton-Vatson branching process. Our main analytical tool is the slow variation (or more general, a regular variation)conception in the sense of Karamata. The slow variation property arises in many issues, but it usually remains rather hidden. Application of Karamata functions in the branching processes theory allows one to bypass severe constraints concerning existence of the higher-order moments of the infinitesimal characteristics of the process under study. In this work,delving deeply in the nature of the Karamata functions, we study more subtle properties of branching processes.

Keywords: discrete-time branching process, transition functions, state space classification, generating functions, slowly varying function.

Introduction

The concept of slow variation was first introduced by Jovan Karamata. Zolotaryov [8] was one of the first to demonstrate the promising prospects of applying this theory in the theory of stochastic branching processes. Later, Slack [2], [3] and Seneta [4], [5], [6] proved new limit theorems in the theory of branching processes by utilizing slowly varying functions. Furthermore, in works [9], [11]–[16], new results in the theory of branching processes were obtained by applying slowly varying functions.

A positive and measurable function $l(x)$ defined on the half-line $[a,\infty)\subset\mathbb{R}_+$ is called slowly varying at infinity if, for any $\lambda>0$ the following condition holds:

$$\frac{l(\lambda x)}{l(x)}=1$$

If the function $r(x)$ can be expressed in the form $r(x)=x^{\rho}l(x)$, it is called a ρ-order regularly varying function, where $l(x)$ is a slowly varying function. Detailed information about these functions can be found in the monographs [8], [10].

Consider a Galton-Watson (G-W) branching process $\{Z(n),\, n\in\mathbb{N}_0=\{0\}\cup\mathbb{N}\}$ with the generating function $f(s)=\sum_{j\in\mathbb{N}_0} p_j s^j$. If the condition $\sum_{j\in\mathbb{N}} jp_j<\infty$ holds, then it follows that $m:=EZ(1)=f'(1)$. Based on the value of the parameter m, we classify the process into different types. If <1 , the process is *subcritical*; if $m=1$, the process is *critical*; and if $m>1$, the process is *supercritical*. In this work, we consider the case when the process is critical.

Main Results

Let us consider the branching law $\{p_k, k\in\mathbb{N}_0\}$ and its generating function in the form of the branching Galton-Watson (G-W) process $\{Z(n), n\in\mathbb{N}_0\}$ given by equation

$f(s)=s+(1-s)^{1+\nu}\mathcal{L}\left(\frac{1}{1-s}\right)$ (1),

where $\nu\in(0,1]$ and $\mathcal{L}(\cdot)$ are slowly varying in infinity. For convenience, in the following discussion, we will use the notation:

$$\Lambda(y):=y^{\nu}\mathcal{L}\left(\frac{1}{y}\right)$$

In this case, we can rewrite the condition (1) in the form of equation

$$f(s)=s+(1-s)\Lambda(1-s). \qquad (2)$$

The expression for the generating function in (1) indicates that the process under consideration is critical, that is, m=1. From this expression, according to the property of slowly varying functions, the second-order moment of the branching law results in

$$f''(1-)=\lim_{s\uparrow 1}\frac{2}{(1-s)^{1-\nu}}\mathcal{L}\left(\frac{1}{1-s}\right)=\lim_{y\downarrow 0}\frac{\Lambda(y)}{y}=\infty.$$

for the case of $\nu\in(0,1)$. It is evident that if $f''(1-)<\infty$ holds, the condition (1) will be satisfied by $\nu=1$ and will result in $2\mathcal{L}(t)\to f''(1-)$, $t\to\infty$. Therefore, condition (1) describes a critical G-W process where the variance of the branching law may be infinite.

Since the process is critical, it is known from the literature that the extinction probability of the considered process is $q=1$ and $n\to\infty$ is uniformly $f_n(s)\to 1$ for all values of $s\in[0,1)$. It is known that this quantity $Q_n=1-f_n(0)$ represents the probability of the process continuing at time n. We seek the asymptotic form of this probability as $n\to\infty$. First, we will prove the following statement.

Theorem 1. *For the critical branching process whose generating function is given in the form of equation (1), the following asymptotic relation holds:*

$$\Lambda(Q_n)\sim\frac{1}{\nu n},\quad n\to\infty. \qquad (3)$$

Proof. By definition, for the transformation $y=1-s$, we have $r(y):=y\Lambda(y)\in\mathcal{R}_{\infty}^{1+\nu}$. Therefore, by Lamperti's theorem [10, 1.11], we have

$$\frac{y\Lambda'(y)}{\Lambda(y)}\to\nu,\quad y\downarrow 0. \qquad (4)$$

Now, let us introduce the notation $\varphi(y):=1-f(1-y)==y-y\Lambda(y)$, and by using the intermediate value theorem, we can write the following relations:

$$\frac{1}{\Lambda(\varphi(y))}-\frac{1}{\Lambda(y)}=\frac{\Lambda(y)-\Lambda(\varphi(y))}{\Lambda(y)\Lambda(\varphi(y))}=$$

$$=\frac{y\Lambda'\left(y-\theta y\Lambda(\varphi(y))\right)}{\Lambda(\varphi(y))},$$

where $\theta\in(0,1)$. The right-hand side of the last equality can be written as

$$\frac{y}{y-\theta y\Lambda(y)}\cdot\frac{\left(y-\theta y\Lambda(y)\right)\Lambda'\left(y-\theta y\Lambda(y)\right)}{\Lambda(\varphi(y))}\cdot\frac{\Lambda\left(y-\theta y\Lambda(y)\right)}{\Lambda(\varphi(y))}$$

Taking the limit as $y\downarrow 0$, in this expression, the first term approaches 1, the second product approaches ν by property (4) and the definition of regularly varying functions, and the third product approaches 1. Therefore,

$$\frac{1}{\Lambda(\varphi(y))}-\frac{1}{\Lambda(y)}\to\nu,\ y\downarrow 0. \qquad \textbf{(5)}$$

In the formula (2) above, if we replace the variable s with Q_n, we obtain the equation

$$Q_{n+1}=Q_n-Q_n\Lambda(Q_n).$$

It is known that as $n\to\infty$, $Q_n\to 0$, and the relation (5) can be applied for $y=Q_n$. Then, we have $\frac{1}{\Lambda(Q_n)}-\frac{1}{\Lambda(y)}=\nu+\delta_n$, where $\delta_n\to 0$, $n\to\infty$. If we sum both sides of this equality with respect to n, we obtain the equation

$$\frac{1}{\Lambda(Q_n)}-\frac{1}{\Lambda(1)}=\nu n+\sum_{k=0}^{n-1}\delta_k. \qquad \textbf{(6)}$$

It is known that $\sum_{k=0}^{n-1}\delta_k=o(n)$, $n\to\infty$ and according to the definition of the G-W process, we have $\Lambda(1)=p_0$. By applying these facts in equation (6), we derive the asymptotic formula (3).

The proof of the theorem is complete.

The following result follows from the proven theorem.

Result. *Under the conditions of Theorem 2.1, the probability of the continuation of the process is given by the following relation*:

$$Q_n=\frac{N(n)}{(\nu n)^{1/\nu}}, \qquad (7)$$

where $N(\cdot)\in S_\infty$ *and* $N^{\nu}(n)\mathcal{L}\left(1/Q_n\right)\to 1,\ \ n\to\infty.$

For the proof, it is sufficient to apply the definition of the function $\Lambda(y)$ in equation (3).

It is worth noting that a result similar to equation (7) was proven by Slack in 1968 [3] under the condition that $\mathcal{L}(\cdot)\in S_0$.

Now, we consider the following functions:

$$R_n(s)=1-f_n(s) \quad \text{and} \quad u_n(s)=\frac{f_n(s)-f_n(0)}{f_n(0)-f_{n-1}(0)}.$$

In Slack's aforementioned work [3], it is proven that the limit function $U(s):=lim_{n\to\infty}u_n(s)$ exists for all $s\in[0,1)$, and it satisfies the Abel equation:

$U(f(s))=U(s)+1.$

It is known that the function $U(s)$ can be expressed as the limit of the series $U(s)=\sum_j u_j s^j$ The Abel equation expresses the invariance property of the coefficients $\{u_j\}$ with respect to the transition probabilities P_{ij}, that is, $u_j=\sum_i u_i P_{ij}(\cdot)$.

Now, by substituting the variable s with $f_n(s)$ in formula (2), we obtain the following equation for all $s\in[0,1)$:

$$R_{n+1}(s)=R_n(s)\left(1-\Lambda\left(R_n(s)\right)\right). \qquad (8)$$

From the derived formula (8), it is clear that as $n\to\infty$ and $s=0$, we have $Q_{n+1}\sim Q_n$. Thus, we have

$$u_n(s)=\frac{Q_n-R_n(s)}{Q_{n-1}\Lambda(Q_{n-1})}\sim U_n(s),\ n\to\infty, \qquad (9)$$

where

$$U_n(s)=\frac{1}{\Lambda(Q_n)}\left[1-\frac{R_n(s)}{Q_n}\right]. \qquad (10)$$

In our last considerations, we utilized the fact that $\Lambda(y)\to 0$ as $y\downarrow 0$. Let us consider the auxiliary function $U_n(s):=n\Lambda(Q_n)U_n(s)$. Then, according to the previously mentioned result of Slack, formula (3), and the relation (9), from equation (10) we can write the following equation:

$$\frac{R_n(s)}{Q_n}=1-\frac{U_n(s)}{vn}. \qquad (11)$$

Finally, from equations (11) and (7), we can write the following confirmation.

Lemma 1. *For critical branching processes with a generating function given in the form (1), the following asymptotic relation holds:*

$$R_n(s)=\frac{N(n)}{(\nu n)^{1/\nu}}\left[1-\frac{U_n(s)}{vn}\right], \qquad (12)$$

where $N(\cdot)\in S_\infty$ *and*

$$N(n)\mathcal{L}^{1/\nu}\left(\frac{(vn)^{1/v}}{N(n)}\right)\to 1,\quad n\to\infty$$

and the function $U_n(s)$ *satisfies the following conditions*:

- $U_n(s)\to U(s),$, as $n\to\infty$, and $U(f(s))=U(s)+1$;
- *for all fixed* $n\in\mathbb{N}$, $\lim_{s\uparrow 1} U_n(s)=\nu n$;
- *for all fixed* $n\in\mathbb{N}$, $U_n(0)=0$.

Remark 2.1. The lemma proven above is an analog of the Fundamental Lemma in the theory of critical branching processes. The difference from the classical analog of Lemma 1 is that formula (12) was proven for the case where the variance of the reproductive law is unknown. For example, in the monograph [1,p.23], the asymptotic expansion of the function $R_n(s)$ was found under the condition $f''(1)<\infty$.

Remark 2.2. If we consider the proof of formula (12) at $s=0$ value, according to the relevant parts of Lemma 1, we derive the formula (7) obtained for the survival probability.

III. Conclusion

In the theory of critical Galton-Watson branching processes, it is impossible to compute the variance without the second-order moment. This problem can be solved using slowly varying functions at infinity. The lemma proven in the article serves as a solution to this problem.

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