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TYPES OF CONVERGENCE OF RANDOM VARIABLES. CONVERGENCE OF DISTRIBUTIONS

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Abstract

In the course of mathematical analysis, various approximations of sequences of functions are considered: smooth approximation, approximation at almost all points, approximation by measure, mean square approximation, and other similar approximations. Similarly, in probability theory, sequences of random variables and, at the same time, various approximations of sequences of distribution functions are considered. These approximations serve as the basis for limit theorems in probability theory.

Let a probability space be given in which random variables (Ω,F,P), $\{\xi_n, n\geq 1\}$ form a sequence and random variable ξ is defined.

Definition 1. The sequence of random variables $\{\xi_n, n\geq 1\}$ is said to converge to ξ in probability if for every $\varepsilon>0$

$P(|\xi_n-\xi|>\varepsilon)\to 0, \quad n\to\infty.$

Convergence in probability is denoted in the form

$\xi_n \xrightarrow{P} \xi, \quad n\to\infty.$

In the above definition it is required that the sequence $\{\xi_n, n\geq 1\}$ and the random variable ξ be defined on the same probability space.

Definition 2. The sequence of random variables $\{\xi_n, n\geq 1\}$ is said to converge almost surely to the random variable ξ if $\xi_n(\omega)\to\xi(\omega)$ the convergence $n\to\infty$ holds at almost every point of $\omega\in\Omega$, i.e., the set of points where convergence fails has probability $N\subset\Omega$ equal to $P(N)=0$.

Here, as in $\xi_n \xrightarrow{P} \xi$, it is also assumed that the sequence of random variables and the limiting random variable are defined on a single probability space.

If $\{\xi_n, n\geq 1\}$ is a sequence of random variables, we consider the event $A=\{\xi_n ketma-ketlik\ \xi ga\ yaqinlashadi\}=\{\lim\xi_n=\xi\}=\{\omega:\xi_n(\omega)\to\xi(\omega)\}$. In fact, to assert that $A\subset\Omega$ is an event it is enough to note that it can be written in the form

$$A=\bigcap_{k=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\left\{\omega:|\xi_n(\omega)-\xi(\omega)|\leq\frac{1}{k}\right\}$$

where the event

$$A_{N,k}=\bigcap_{n=N}^{\infty}\left\{|\xi_n-\xi|\leq\frac{1}{k}\right\}$$

that the inequality $|\xi_n-\xi|\leq\frac{1}{k}$ holds for all $n\geq N$ is

$B_k=\bigcup_{N=1}^{\infty}A_{N,k}$,

the event that there exists a natural number N such that the inequality $|\xi_n-\xi|\leq\frac{1}{k}$ holds for $n\geq N$ is

$\bigcap_{k=1}^{\infty}B_k$,

and the event that for every natural number $N>1$ there exists $n\geq N$ such that $|\xi_n-\xi|\leq\frac{1}{k}$ holds, i.e. shows that $\lim\xi_n=\xi$ occurs.

Based on the above, convergence with probability 1 can be defined as follows.

Definition 2'. If

$$P(A)=P(\lim\xi_n=\xi)=P\left(\bigcap_{k=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\left\{|\xi_n-\xi|\le\frac{1}{k}\right\}\right)=1$$

then the sequence of random variables $\{\xi_n,\ n\ge1\}$ is said to converge with probability 1 to the random variable ξ, and it is denoted by

$P(\xi_n\to\xi)=1$ or $P(\lim\xi_n=\xi)=1$.

The following not-too-complicated remarks clarify the nature of convergence with probability 1.

If

$$P\left(\bigcap_{k=1}^{\infty}B_k\right)=1$$

then for all k we have $P(B_k)=1$, and conversely if for all k we have $P(B_k)=1$, $P(\cap_{k=1}^{\infty}B_k)=1$.

Hence, for convergence with probability 1 to $\xi_n\to\xi$,

$$P\left(\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\left\{|\xi_n-\xi|\le\frac{1}{k}\right\}\right)=1$$

the equality must hold; it is both necessary and sufficient. Since the sequence of events in the above

$$A_{N,k}=\bigcap_{n=N}^{\infty}\left\{|\xi_n-\xi|\le\frac{1}{k}\right\}$$

is monotonically decreasing $(A_N\subseteq A_{N+1})$, by the continuity property of probability, for (1) to hold it is necessary and sufficient that the relation

$$\lim_{N\to\infty}P(A_{N,k})=\lim_{N\to\infty}P\left(\bigcap_{n=N}^{\infty}\left\{|\xi_n-\xi|\le\frac{1}{k}\right\}\right)=1$$

hold for all k.

In the last relation $\frac{1}{k}$ may be replaced by any $\varepsilon>0$.

Thus the following theorem is valid.

Theorem 1. For the sequence of random variables $\{\xi_n,n\ge1\}$ to converge with probability 1 to the random variable ξ it is necessary and sufficient that relation

$$\lim_{N\to\infty}P\left(\bigcap_{n=N}^{\infty}\{|\xi_n-\xi|\le\varepsilon\}\right)=1$$

holds (for every $\varepsilon>0$).

Remark. From the validity of the inequality

$$P(|\xi_n-\xi|\le\varepsilon)\ge P\left(\bigcap_{k=n}^{\infty}\{|\xi_n-\xi|\le\varepsilon\}\right)$$

and the fact that ξ_n converges to ξ with probability 1, it follows that it converges to ξ in probability, because in this case

$P(|\xi_n-\xi|>\varepsilon)=1-P(|\xi_n-\xi|\le\varepsilon)\to0$, $n\to\infty$.

Corollary 1. The condition of Theorem 1 is equivalent to the equality

$$\lim_{N\to\infty}P(\cup_{n=N}^{\infty}\{|\xi_n-\xi|>\varepsilon\})=0 \qquad (1)$$

for any $\varepsilon>0$ being satisfied.

Now we introduce the following sequence of random variables

$\zeta_n=\sup_{k\geq n}|\xi_k-\xi|$.

In this case Theorem 1 can be written in the following form:

Theorem 2. For the sequence $\{\xi_n,\ n\geq 1\}$ to converge with probability 1 to ξ it is necessary and sufficient that the relations $\zeta_n \xrightarrow{P} 0,\ \ n\to\infty$ hold.

Indeed,

$$\{\zeta_n>\varepsilon\}=\cup_{k=n}^{\infty}\{|\xi_k-\xi|>\varepsilon\} \qquad (2)$$

the equality holds, and the proof of Theorem 2 follows from relations (1) and (2).

Above we saw that from a limit relation $\left(\xi_n \xrightarrow{P} \xi\right)$ the equality $P(\lim\xi_n=\xi)=1$ does not necessarily follow. But if the probability

$$P(|\xi_n-\xi|>\varepsilon)$$

tends to zero at a known rate, then the sequence $\{\xi_n,n\geq 1\}$ may converge with probability 1 to ξ.

Theorem 3. If for every $\varepsilon>0$

$$\sum_{n=1}^{\infty}P(|\xi_n-\xi|>\varepsilon)<\infty \qquad (3)$$

then $P(\lim\xi_n=\xi)=1$.

The proof of this theorem follows from

$$P\left(\bigcup_{n=N}^{\infty}\{|\xi_n-\xi|>\varepsilon\}\right)\leq\sum_{n=N}^{\infty}P(|\xi_n-\xi|>\varepsilon)$$

and equality (1). But from relation (3) one cannot conclude how fast the probability $P(|\xi_n-\xi|>\varepsilon)$ tends to zero.

From the stated Theorem 3 we obtain the following result.

Corollary 2. If $\xi_n \xrightarrow{P} \xi$, then there exists a subsequence $\{\xi_{n_k},k\geq 1\}$ such that for it $P(\lim\xi_{n_k}=\xi)=1$.

The proof of this result is easy because one can choose indices as n_k so that for them the inequality

$$P(|\xi_n-\xi|>\varepsilon)\leq\frac{1}{k^2}$$

holds.

The condition (3) of Theorem 3 is a basic criterion in probability theory for studying properties that are described with probability 1; we state the following Borel–Cantelli lemma.

Let

$$A_1,A_2,\ldots,A_n,\ldots$$

be a sequence of events. The event that infinitely many of these events occur is

$$\limsup A_n=\overline{\lim}A_n=\{A_n,\ cheksiz\ ko'p\ n\ uchun\}.$$

Borel–Cantelli lemma. The following relations hold:

a) If $\sum_{n=1}^{\infty}P(A_n)<\infty$ then $P(A_n,\ ch.k)=0$.

b) If $\sum_{n=1}^{\infty}P(A_n)=\infty$ and the events $A_1,A_2,\ldots$ are independent then $P(A_n,\ ch.k.)=1$.

Proof.

a) By definition,

$$\{A_n,ch.k\}=\{\overline{\lim}A_n\}=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k$$

and by the continuity property of probability $P(A_n,ch.k)$,

$$P(A_n,ch.k)=P(\cap_{n=1}^{\infty}\cup_{k=n}^{\infty}A_k)=\lim_{n\to\infty}P(\cup_{k=n}^{\infty}A_k)\leq$$

$$\leq\lim_{n\to\infty}\sum_{k=n}^{\infty}P(A_k).$$

From the last relation conclusion a) follows.

b) If $A_1,A_2,\ldots$ the events are independent, then $\overline{A}_1,\overline{A}_2,\ldots$ events are also independent. In this case for any $N\geq n$

$$P\left(\bigcap_{k=n}^{N}\overline{A}_k\right)=\prod_{k=n}^{N}P(\overline{A}_k)$$

and from this equality it is not hard to see that

$$P\left(\cap_{k=n}^{\infty}\overline{A}_k\right)=\prod_{k=n}^{\infty}P(\overline{A}_k) \qquad (4)$$

holds.

Here, using the elementary inequality

$\log(1-x)\leq -x,\ \ 0\leq x<1,$

$\log\prod_{k=n}^{\infty}\left(1-P(A_k)\right)=\sum_{k=n}^{\infty}\log[1-P(A_k)]\leq 104$

$\leq-\sum_{k=n}^{\infty}P(A_k)=\ \ -\infty.$

Hence, for any n

$$P\left(\bigcap_{k=n}^{\infty}\overline{A}_k\right)=0$$

and $P(A_n,ch.k)=1$.

The lemma is proved.

Corollary 3. If $A_n^{\varepsilon}=\{\omega;\ |\xi_n(\omega)-\xi(\omega)|<\varepsilon\}$, then under condition (3) for any $\varepsilon>0$

$$\sum_{n=1}^{\infty}P(A_n^{\varepsilon})<\infty,\varepsilon>0$$

the series converges.

Thus, according to the Borel–Cantelli lemma,

$P(A^{\varepsilon})=P\left(\overline{\lim}A_n^{\varepsilon}\right)=0,\ \ \varepsilon>0.$

From this the following logical implications hold:

$$\sum_{n=1}^{\infty}P(|\xi_n-\xi|\geq\varepsilon)<\infty,\quad \varepsilon>0\Rightarrow$$

$$\Rightarrow P(A^{\varepsilon})=0,\varepsilon>0\Leftrightarrow P(\xi_n\nrightarrow\xi)=0.$$

Corollary 4. Let positive sequences $\varepsilon_n\downarrow 0$, and $n\to\infty$ be given. If

$$\sum_{n=1}^{\infty}P(|\xi_n-\xi|\geq\varepsilon_n)<\infty$$

then $P(\xi_n\to\xi)=1$.

Indeed,

$$A_n=\{|\xi_n-\xi|\geq\varepsilon_n\}$$

let. By the Borel–Cantelli lemma, $P(A_n\ ch.k)=0$. According to the last equality, for almost every elementary outcome $\omega\in\Omega$ there exists a natural number $N=N(\omega)$ such that $n\geq N(\omega)$ whenever $|\xi_n-\xi|\leq\varepsilon_n$. But $\varepsilon_n\downarrow 0$ implies that $\xi_n(\omega)\to\xi(\omega)$ for almost all $\omega\in\Omega$.

Above we have studied three types of convergence of random variables defined on a single probability space (Ω,F,P) (convergence with probability 1, in probability, and in the (r)-th mean). But if random variables are defined on different probability spaces (or their underlying spaces are not specified) , and their distributions are "similar", the question naturally arises how to understand the "convergence" of such random variables. In this situation it is important to express convergence of random variables via properties of convergence of their distributions. If we can find convenient properties of convergence of distributions, we gain the ability to approximate complicated distributions (which are needed but hard to compute) by simpler ones. These ideas form the basis of the section of modern probability theory called "limit theorems

Definition 3. The sequence of random variables $\{\xi_n,\ n\geq 1\}$ is said to converge to random variable ξ in $r-$th mean if

$$E|\xi_n-\xi|^r\to 0,\quad n\to\infty$$

holds. This convergence is denoted by

$\xi_n \overset{(r)}{\to} \xi, n\to\infty$.

By Chebyshev's inequality,

$$P(|\xi_n-\xi|\geq\varepsilon)\leq\frac{E|\xi_n-\xi|^r}{\varepsilon^r}$$

Hence, convergence in $r-$th mean implies convergence in probability $\left(\left(\xi_n \overset{(r)}{\to} \xi\right)\Rightarrow\left(\xi_n \overset{p}{\to} \xi\right)\right)$.

However, convergence in probability does not imply convergence in mean.

The following theorem studies under what conditions convergence with probability 1 implies convergence in the first mean $\xi_n \overset{(1)}{\to} \xi$.

Theorem 4. Let $\{\xi_n,\ n\geq 1\}-$ be a sequence of nonnegative random variables such that

$P(\xi_n\to\xi)=1,\quad E\xi_n\to E\xi<\infty$.

Then $E|\xi_n-\xi|\to 0,\quad n\to\infty$.

Proof. Under the given conditions, for sufficiently large n we have $E\xi_n<\infty$. Therefore

$$E|\xi_n-\xi|=E|\xi-\xi_n|=E(\xi-\xi_n)I_{\{\xi\geq\xi_n\}}+$$
$$+E(\xi_n-\xi)I_{\{\xi_n>\xi\}}=2E(\xi-\xi_n)I_{\{\xi\geq\xi_n\}}+E(\xi_n-\xi). \qquad (5)$$

But

$0\leq(\xi-\xi_n)I_{\{\xi\geq\xi_n\}}\leq\xi$.

From the last and the fact that $P(\xi_n\to\xi)=1$ we obtain

$$\lim_{n\to\infty}(\xi-\xi_n)I_{\{\xi\geq\xi_n\}}=0. \qquad (6)$$

Now the proof of Theorem 4 follows from relations (5) and (6).

Remark. In (6) the condition $P(\xi_n\to\xi)=1$ may be replaced by convergence in probability $\left(\xi_n \overset{p}{\to} \xi\right)$ and the statement remains valid. Below we define F_n to be convergence to F in such a way as to account for situations where the corresponding random variables are close in distribution while being close in probability.

Definition 4. A sequence of distribution functions $\{F_n(x), n\geq 1\}$ is said to converge weakly to the distribution function $F(x)$ if for every bounded continuous function $f(x)$

$$\int_{-\infty}^{\infty} f(x)\,dF_n(x)\to\int_{-\infty}^{\infty} f(x)\,dF(x) \qquad (7)$$

the convergence holds.

Weak convergence is denoted by $F_n\Rightarrow F$.

If $P_n(B)$ and $P(B)$ are probability measures corresponding to distribution functions $F_n(x)$ and $F(x)$ (B are Borel sets), then we say that measure P_n converges weakly to measure P and denote it by $P_n\Rightarrow P$.

In this case (7) can be written in the following form:

$$\int_R f(x)P_n(dx)\to\int_R f(x)P(dx),$$

or

$Ef(\xi_n)\to Ef(\xi),\quad n\to\infty$,

where $P(\xi_n\in B)=P_n(B),\quad P(\xi\in B)=P(B),\quad B-$ is a Borel set.

Another variant of weak convergence follows from the next theorem.

Theorem 5. Weak convergence $P_n\Rightarrow P$ holds if and only if the convergence $F_n(x)\Rightarrow F(x)$ holds at every continuity point $F(x)$ of x.

Remark 1. Based on Theorem 5, the weak convergence of distribution functions can equivalently be defined in the following strong form: distribution function $F_n(x)$ converges weakly to $F(x)$ if $F_n(x) \Rightarrow F(x)$ for every $x \in C(F)$. However, variant (7) has many advantages; for example, it is applicable to probability distributions in infinite-dimensional spaces.

Remark 3. If $F(x)$ is a continuous-type distribution function, then convergence $F_n \Rightarrow F$ is equivalent to uniform convergence

$$\sup_x |F_n(x) - F(x)| \to 0, n \to \infty.$$

The proof of this statement $F_n(x) - F(x)$ follows from the fact that the convergence is uniform on any finite interval because the limit is continuous $F(x)$.

Remark 4. If distributions F_n and F are discrete with jumps at common points $x_1, x_2, \ldots, x_n, \ldots$, then $F_n \Rightarrow F$

$$F_n(x_k + 0) - F_n(x_k) \to F(x_k + 0) - F(x_k)$$

is equivalent to convergence. In the latter, convergence of probabilities corresponding to the values $x_1, x_2, \ldots, x_n, \ldots$ is written.

Let a sequence of probability distributions $\{P_n,\ n \geq 1\}$ be given and let the corresponding random variables sequence be

$$\xi_1, \xi_2, \ldots, \xi_n, \ldots$$

such that for them

$P_n(B) = P(\xi_n \in B),\quad n = 1,2,\ldots,$

B–are Borel sets. In general, it is not required that these random variables be defined on a single probability space.

Definition 5. If $P_n \Rightarrow P,\ P(\xi \in B) = P(B)$ then we say that the sequence of random variables $\{\xi_n,\ n \geq 1\}$ converges in distribution to ξ and denote it by $\xi_n \xrightarrow{d} \xi$.

Often convergence in distribution of random variables is called convergence by distribution.

Clearly, if $\xi_n \xrightarrow{p} \xi$ then $\xi_n \xrightarrow{d} \xi$. But $\left(\xrightarrow{d}\right)$ does not imply $\left(\xrightarrow{p}\right)$. The following theorem clarifies this situation.

Theorem 6. If $\xi_n \xrightarrow{d} \xi$ $(F_n \Rightarrow F)$, then one can construct random variables ξ'_n and ξ' on a single probability space such that

$$P(\xi'_n < x) = P(\xi_n < x) = F_n(x)$$
$$P(\xi' < x) = P(\xi < x) = F(x)$$

and

$$P(\xi'_n \to \xi') = 1$$

i.e., the random variables ξ'_n converge with probability 1 to ξ'.

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